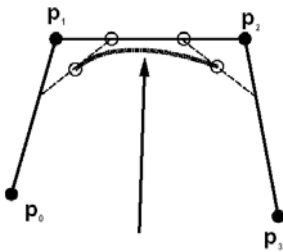


# B-Spline Curves and NURBS

# Motivation

- The goal is local control
- B-splines provide local control
- Do not interpolate control points.
- $C^2$  continuity

# B-spline Curve



# B-Splines

- Curve:  $c(u) = \sum_{i=0}^n p_i B_{i,k}(u)$
- Control points  $p_i$
- Basis functions of degree  $k-1$ :  $B_{i,k}(u)$
- How to formulate  $B_{i,k}(u)$
- First we introduce a knot sequence  $(n+k+1)$  in a non-decreasing order

$$u_0, u_1, u_2, \dots, u_{n+k}$$

# B-Splines Basis Functions: Cox-deBoor recursion

- The basis function are defined recursively:

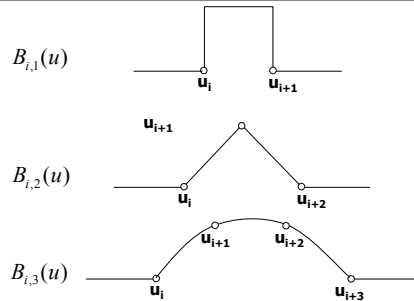
$$B_{i,1}(u) = \begin{cases} 1 & u_i \leq u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{i,k}(u) = \frac{u - u_i}{u_{i+k-1} - u_i} B_{i,k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} B_{i+1,k-1}(u)$$

- The parametric domain of B-spline is :

$$u \text{ in } [u_{k-1}, u_{n+1}]$$

# B-Spline Basis Functions



## B-Spline Basis Functions

- Linear Basis Functions:
- Knot vector: [0, 1, 2, 3, 4]

$$B_{0,2}(u) = \begin{cases} u & u \in [0, 1] \\ 2 - u & u \in [1, 2] \end{cases}$$

$$B_{1,2}(u) = \begin{cases} u - 1 & u \in [1, 2] \\ 3 - u & u \in [2, 3] \end{cases}$$

$$B_{2,2}(u) = \begin{cases} u - 2 & u \in [2, 3] \\ 4 - u & u \in [3, 4] \end{cases}$$

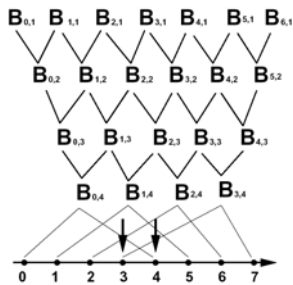
## B-Spline Basis Functions

- Quadratic Basis Functions:
- Knot vector: [0, 1, 2, 3, 4, 5, 6]

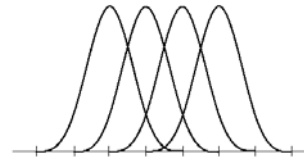
$$B_{0,3}(u) = \begin{cases} \frac{1}{2}u^2 & u \in [0, 1] \\ \frac{1}{2}u(2-u) + \frac{1}{2}(3-u)(u-1) & u \in [1, 2] \\ \frac{1}{2}(3-u)^2 & u \in [2, 3] \end{cases}$$

$$B_{1,3}(u) = \begin{cases} \frac{1}{2}(u-1)^2 & u \in [1, 2] \\ \frac{1}{2}(u-1)(3-u) + \frac{1}{2}(4-u)(u-2) & u \in [2, 3] \\ \frac{1}{2}(4-u)^2 & u \in [3, 4] \end{cases}$$

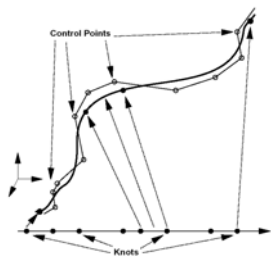
## B-Spline Basis Function



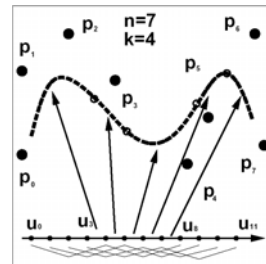
## B-Spline Basis Functions



## B-Splines



## B-Splines



## B-Splines Properties

- $C^2$  continuity.
- Approximation.
- Local control.
- Each segment is determined by four control points.
- Convex hull
- What if we put more than one control points in the same location?
  - Double vertices
  - Triple vertices

## B-Spline Properties

- Double endpoint:
  - Curve will be tangent to line between first distinct points.
- Triple endpoint:
  - Curve interpolate endpoint, start with a line segment.

## Nonuniform B-Splines

- Multiple knots have the effect of pulling the spline closer to the control point associated with the knot.
- If a knot at the end has multiplicity  $d+1$ , the B-spline of degree  $d$  must interpolate the point.
- The knot sequence  $\{0,0,0,0,1,2,\dots,n-1,n,n,n,n\}$  is often used for cubic B-splines. In particular, if the knot sequence is  $\{0,0,0,0,1,1,1,1\}$ , then B-splines becomes the cubic Bezier curve.

## B-Spline Rendering

- Transform it to a set of Bezier curves.
- Display B-Spline ( $\{p_0, \dots, p_n\}$ )
- For  $i=0$  to  $i=n-k+1$  do
- Convert  $i$ -span controlled by  $p_i, \dots, p_{i+k-1}$  into a bezier curve represented by  $v_0, \dots, v_{k-1}$
- Display Bezier Curve ( $\{v_0, \dots, v_{k-1}\}$ )

## From B-Splines to Bezier



## From B-Splines to Bezier

- B Splines control points:  $p_0, p_1, p_2, p_3$
- Bezier curves control points:  $v_0, v_1, v_2, v_3$ .

$$v_1 = \frac{2p_{i+1} + p_{i+2}}{3}$$

$$v_2 = \frac{p_{i+1} + 2p_{i+2}}{3}$$

$$v_0 = \frac{1}{6}(p_i + 4p_{i+1} + p_{i+2})$$

$$v_3 = \frac{1}{6}(p_{i+1} + 4p_{i+2} + p_{i+3})$$

## From B-Splines to Bezier

- In matrix form:

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} P_i \\ P_{i+1} \\ P_{i+2} \\ P_{i+3} \end{bmatrix}$$

## From B-Splines to NURBS

- What are NURBS??
- Non Uniform Rational B-Splines (NURBS)
- Why rational curves?
- Polynomial-based splines can not represent commonly-used analytic shapes such as conic section, circles, ellipses, parabolas.
- Rational Splines can achieve this goal.
- NURBS are a unified representation
  - Polynomial, conic section, etc.
  - Industry standard.

## NURBS

- NURBS mathematics:

$$c(u) = \frac{\sum_{i=0}^n P_i w_i B_{i,k}(u)}{\sum_{i=0}^n w_i B_{i,k}(u)}$$

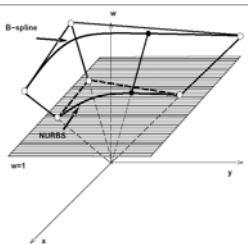
- Geometric Meaning--- obtained from projection!
- B-splines in homogenous representation

$$c(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \\ w(u) \end{bmatrix} = \sum_{i=0}^n \begin{bmatrix} P_{i,x} w_i \\ P_{i,y} w_i \\ P_{i,z} w_i \\ w_i \end{bmatrix} B_{i,k}(u) = \sum_{i=0}^n \begin{bmatrix} P_i w_i \\ w_i \end{bmatrix} B_{i,k}(u)$$

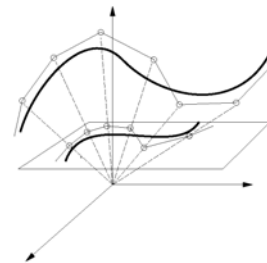
## Properties of NURBS

- Represent standard shapes.
- Invariant under perspective projection.
- B-Spline is a special case.
- Weights as extra degrees of freedom.
- Can represent analytic shapes such as circles.

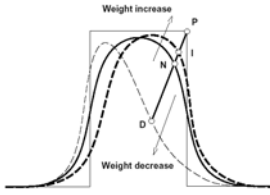
## NURBS



## NURBS



# NURBS



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# NURBS



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