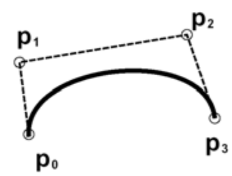
Bezier Curves

Bezier Curve



Cubic Bezier Curves

- Curve: $c(u) = \sum_{i=0}^{3} p_i B_i^3(u)$
- Control points p_i
- Basis functions:

$$B_0^3(u) = (1-u)^3$$
$$B_1^3(u) = 3u(1-u)^2$$
$$B_2^3(u) = 3u^2(1-u)$$
$$B_3^3(u) = u^3$$

Bezier Curves (degree n)

- Curve: $c(u) = \sum_{i=0}^{n} p_i B_i^n(u)$
- Control points p_i
- Basis functions $B_i^n(u)$ are bernstein polynomials of degree n:

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}$$

$$\binom{n}{i} = \frac{n!}{(n-i)!\,i!}$$

Properties

- End point interpolation.
- Basis functions are non-negative.
- The summation of basis functions are unity
  - Binomial Expansion Theorem:

$$1 = [u + (1-u)]^n = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i}$$

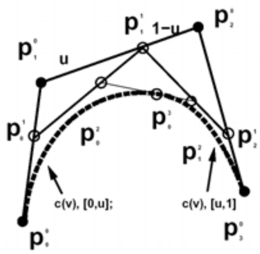
- Convex hull: the curve is bounded by the convex hull defined by the control points.

Recursive Computation: The De Casteljau Algorithm

$$B_i^n(u) = (1-u)B_i^{n-1}(u) + uB_{i-1}^{n-1}(u)$$

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}$$
$$= \binom{n-1}{/i} u^i (1-u)^{n-i} + \binom{n-1}{i-1} u^i (1-u)^{n-i}$$
$$= (1-u)B_i^{n-1}(u) + uB_{i-1}^{n-1}(u)$$

Recursive Computation



Recursive Computation

$$\mathbf{p}_i^0 = \mathbf{p}_i,\ i = 0, \ldots, n$$

$$\mathbf{p}_i^j = (1-u)\mathbf{p}_i^{j-1} + u\mathbf{p}_{i+1}^{j-1}$$

$$\mathbf{c}(u) = \mathbf{p}_0^n(u)$$

Tangents and Derivatives

End-point tangents:

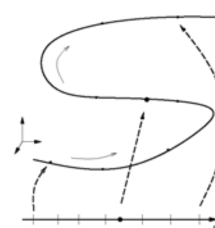
$$\mathbf{c}'(0) = n(\mathbf{p}_1 - \mathbf{p}_0)$$

$$\mathbf{c}'(1) = n(\mathbf{p}_n - \mathbf{p}_{n-1})$$

i-th derivatives:
$\mathbf{c}^{(i)}(0)$ **depends only on** $\mathbf{p}_0, \ldots, \mathbf{p}_i$
$\mathbf{c}^{(i)}(1)$ **depends only on** $\mathbf{p}_n, \ldots, \mathbf{p}_{n-i}$

Derivatives at non-end-points:
$\mathbf{c}^{(i)}(u)$ **involve all control points**

Piecewise Polynomials



Piecewise Bezier Curves

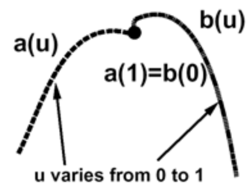
Piecewise polynomials

- Different polynomials for different parts of the curve.
- Advantages: flexible, lower degree.
- Disadvantages: how to ensure smoothness at the joints (continuity).

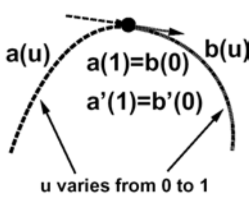
Continuity

- One of the fundamental concepts.
- Commonly used cases: C^0, C^1, C^2, etc.
- C^0 Continuity: Position.
- C^1 Continuity: Velocity.
- C^2 Continuity: Acceleration.

Positional Continuity: C^0



Derivative Continuity: C^1



General Continuity: C^n

- C^n continuity: derivatives (up to the n-th) are the same at end points:
  $a^i(1) = b^i(0)$, where $i=0,1, \ldots, n$.
- This definition is for parametric continuity.
- Parametric continuity depends on parameterization.
- Parameterization is not unique, different parameterization can represent the same geometry.
- Re-parameterization can be easily implemented.

- Another type of continuity: Geometric continuity: G^n

Geometric Continuity: G^n

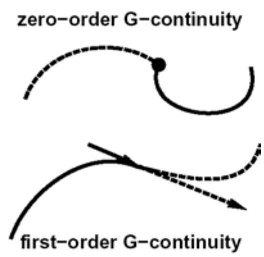
- Depend on the curve geometry
- Do not depend on the underlying parameterization.
- G^0 : the same joint.
- G^1 :Two curve tangents at the joint align, however, may not have the same magnitude.
- G^1 : C^1 it is after the reparameterization.
- Which condition is stronger?

Continuity

- In general, parametric continuity implies geometric continuity.

- Special cases:

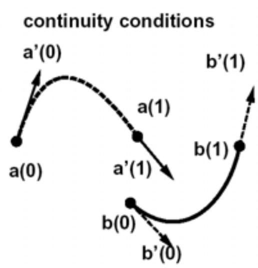
Geometric Continuity



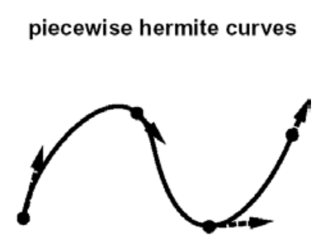
Piecewise Hermite Curves

- How to build an interactive system to satisfy various constraints.
- C^0 continuity:
  - $a(1) = b(0)$
- C^1 continuity:
  - $a(1) = b(0)$
  - $a'(1) = b'(0)$
- G^1 continuity:
  - $a(1) = b(0)$
  - $a'(1) = \alpha b'(0)$

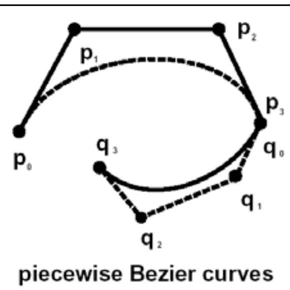
Piecewise Hermite Curves



Piecewise Hermite Curves



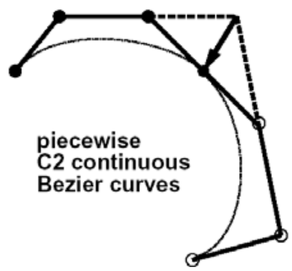
Piecewise Bezier Curves



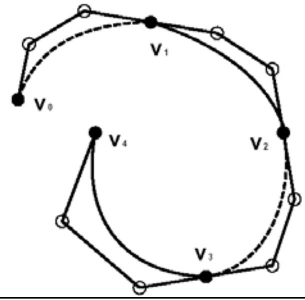
Piecewise Bezier Curves

- C^0 continuity:
  - $p_3 = q_0$
- C^1 continuity:
  - $p_3 = q_0$
  - $p_3 - p_2 = q_1 - q_0$
- G^1 continuity:
  - $p_3 = q_0$
  - $p_3 - p_2 = \alpha(q_1 - q_0)$
- C^2 continuity:
  - $p_3 = q_0$
  - $p_3 - p_2 = q_1 - q_0$
  - $p_3 - 2p_2 + p_1 = q_2 - 2q_1 + q_0$

Piecewise C^2 Bezier Curves



C^2 Interpolating Splines



C^2 Interpolating Splines

- Interpolate all control points
- Equivalent to a thin strip of metal in a physical sense.
- Forced to pass through a set of desired points.
- Advantages:
  - interpolation,
  - C^2
- Disadvantages:
  - No local control (if one point is changes, the entire curve will move)
- How to overcome the drawbacks: B-splines.