

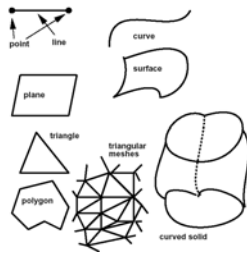
Geometric Modeling

Overview

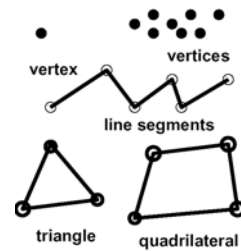
3D Shape Primitives:

- Points
 - Vertices.
- Curves
 - Lines, polylines, curves.
- Surfaces
 - Triangle meshes, splines, subdivision surfaces, implicit surfaces, particles.
- Solids

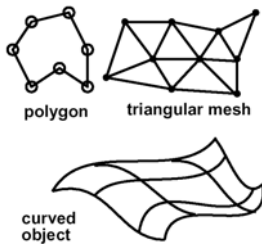
Basic Shapes



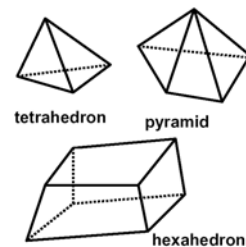
Fundamental Shapes



Fundamental Shapes



Fundamental Shapes



Curves

- Lines.
- Polynomials.
- Lagrange curves.
- Hermite curves.
- Bezier curves.
- B-Splines.
- NURBS.
- Subdivision schemes.

Mathematical Formulations

Point:

$$p = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Line:

$$l(u) = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} u + \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Quadratic curve:

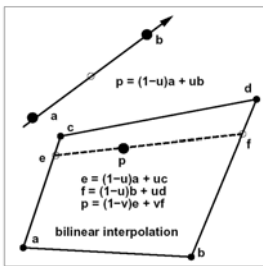
$$q(u) = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} u^2 + \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} u + \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

parametric domain: $u \in [u_1, u_2]$

Reparameterization: $v = (u - u_1)/(u_2 - u_1)$, $v \in [0, 1]$

High-order polynomials

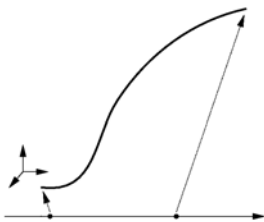
Linear interpolation



Surfaces

- Planes
- Triangle meshes.
- Tensor-product surfaces.
 - Hermite surface, Bezier surface, B-spline surfaces, NURBS.
- No-tensor product surfaces.
 - Sweeping surface, ruling surface, etc.
- Subdivision surfaces.
- Implicit surfaces.
- Particle systems.

Parametric Polynomials



Polynomials

$$c(u) = \begin{bmatrix} a_{0,x} \\ a_{0,y} \\ a_{0,z} \end{bmatrix} + \dots + \begin{bmatrix} a_{i,x} \\ a_{i,y} \\ a_{i,z} \end{bmatrix} u^i + \dots + \begin{bmatrix} a_{n,x} \\ a_{n,y} \\ a_{n,z} \end{bmatrix} u^n$$

- No intuitive insight.
- Difficult for piecewise smooth curves.

Cubic Polynomials

$$\begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} u^3 + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} u^2 + \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} u + \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$$

$u \in [0, 1]$

Cubic Polynomials

$$x(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = UA$$

$$y(u) = UB$$

$$z(u) = UC$$

How to define a curve?

- Specify a set of points for interpolation and/or approximation for some u_i

$$\begin{bmatrix} x(u_i) \\ y(u_i) \\ z(u_i) \end{bmatrix}$$

- Specify the derivatives at some locations for some u_i

$$\begin{bmatrix} x'(u_i) \\ y'(u_i) \\ z'(u_i) \end{bmatrix}$$

- A combination of both.

Lagrange Curve

- The curve interpolates all the control points.
- Unwanted oscillation.



Lagrange Curve

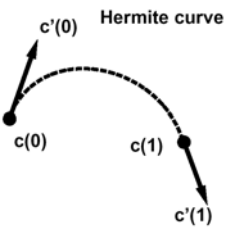
$$c(u) = \begin{bmatrix} a_{n,x} \\ a_{n,y} \\ a_{n,z} \end{bmatrix} L_0^n(u) + \dots + \begin{bmatrix} a_{n,x} \\ a_{n,y} \\ a_{n,z} \end{bmatrix} L_n^n(u)$$

$$L_i^n(u) = \frac{\prod_{j=0, j \neq i}^n (u - u_j)}{\prod_{j=0, j \neq i}^n (u_i - u_j)}$$

Kronecker-delta function

$$L_i^n(u_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Cubic Hermite Curve



Cubic Polynomials

$$\begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} u^3 + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} u^2 + \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} u + \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$$

$$u \in [0, 1]$$

$$x(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = UA$$

$$y(u) = UB$$

$$z(u) = UC$$

Cubic Hermite Curve

- Constraints: two end-points and two tangents at end-points.

$$\begin{bmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} A$$

Cubic Hermite Curve

- Solve the linear equation:

$$A = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{bmatrix}$$

Cubic Hermite Curve

$$x(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = UA$$

$$A = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{bmatrix}$$

Cubic Hermite Curve

$$c(u) = c(0)f_1(u) + c(1)f_2(u) + c'(0)f_3(u) + c'(1)f_4(u)$$

$$f_1(u) = 2u^3 - 3u^2 + 1$$

$$f_2(u) = -2u^3 + 3u^2$$

$$f_3(u) = u^3 - 2u^2 + u$$

$$f_4(u) = u^3 - u^2$$

Why Cubic Curves

- The lowest-degree polynomials to be able to interpolate two specified endpoints with specified derivatives at each endpoint.
- The lowest-degree curves that are nonplanar in 3D.
- Lower degree has too little flexibility.
- Higher-degree is unnecessarily complex, exhibit undesired wiggles.