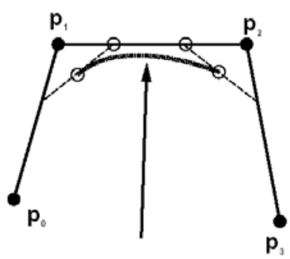
B-Spline Curves and NURBS

Motivation

- The goal is local control
- B-splines provide local control
- Do not interpolate control points.
- C^2 continuity

B-spline Curve



B-Splines

- Curve: $c(u) = \sum_{i=0}^{n} p_i B_{i,k}(u)$
- Control points p_i
- Basis functions of degree k-1: $B_{i,k}(u)$
- How to formulate $B_{i,k}(u)$
- First we introduce a knot sequence (n+k+1) in a non-decreasing order

  $u_0, u_1, u_2, \ldots, u_{n+k}$

B-Splines Basis Functions: Cox-deBoor recursion

- The basis function are defined recursively:

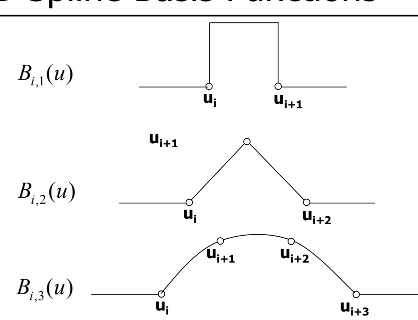
$$B_{i,1}(u) = \begin{cases} 1 & u_i \le u < u_{i+1} \\ 0 & otherwise \end{cases}$$

$$B_{i,k}(u) = \frac{u - u_i}{u_{i+k-1} - u_i} B_{i,k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} B_{i+1,k-1}(u)$$

- The parametric domain of B-spline is :

  u in $[u_{k-1}, u_{n+1}]$

B-Spline Basis Functions



B-Spline Basis Functions

- Linear Basis Functions:
- Knot vector: [0, 1, 2, 3, 4]

$$B_{0,2}(u) = \begin{cases} u & u \in [0,1] \\ 2-u & u \in [1,2] \end{cases}$$

$$B_{1,2}(u) = \begin{cases} u-1 & u \in [1,2] \\ 3-u & u \in [2,3] \end{cases}$$

$$B_{2,2}(u) = \begin{cases} u-2 & u \in [2,3] \\ 4-u & u \in [3,4] \end{cases}$$

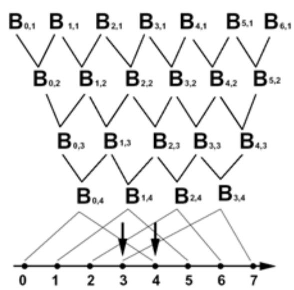
B-Spline Basis Functions

- Quadratic Basis Functions:
- Knot vector: [0, 1, 2, 3, 4, 5, 6]

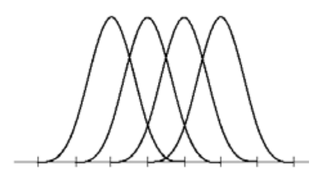
$$B_{0,3}(u) = \begin{cases} \frac{1}{2}u^2 & u \in [0,1] \\ \frac{1}{2}u(2-u) + \frac{1}{2}(3-u)(u-1) & u \in [1,2] \\ \frac{1}{2}(3-u)^2 & u \in [2,3] \end{cases}$$

$$B_{1,3}(u) = \begin{cases} \frac{1}{2}(u-1)^2 & u \in [1,2] \\ \frac{1}{2}(u-1)(3-u) + \frac{1}{2}(4-u)(u-2) & u \in [2,3] \\ \frac{1}{2}(4-u)^2 & u \in [3,4] \end{cases}$$

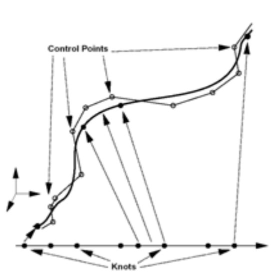
B-Spline Basis Function



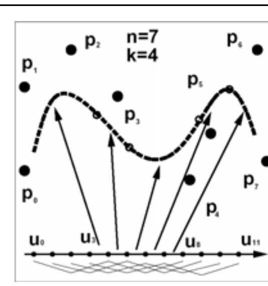
B-Spline Basis Functions



B-Splines



B-Splines



B-Splines Properties

- C^2 continuity.
- Approximation.
- Local control.
- Each segment is determined by four control points.
- Convex hull
- What if we put more than one control points in the same location?
  - Double vertices
  - Triple vertices

B-Spline Properties

- Double endpoint:
  - Curve will be tangent to line between first distinct points.
- Triple endpoint:
  - Curve interpolate endpoint, start with a line segment.

Nonuniform B-Splines

- Multiple knots have the effect of pulling the spline closer to the control point associated with the knot.
- If a knot at the end has multiplicity d+1, the B-spline of degree d must interpolate the point.
- The knot sequence {0,0,0,0,1,2,…,n-1,n,n,n,n} is often used for cubic B-splines. In particular, if the knot sequence is {0,0,0,0,1,1,1,1}, then B-splines becomes the cubic Bezier curve.

B-Spline Rendering

- Transform it to a set of Bezier curves.
- Display B-Spline ($\{p_0, \ldots, p_n\}$)
- For i=0 to i=n-k+1 do
- Convert i-span controlled by $p_i, \ldots, p_{i+k-1}$ into a bezier curve represented by $v_0, \ldots, v_{k-1}$
- Display Bezier Curve ($\{v_0, \ldots, v_{k-1}\}$)

From B-Splines to Bezier

From B-Splines to Bezier

- B Splines control points: p_0, p_1, p_2, p_3
- Bezier curves control points: v_0, v_1, v_2, v_3.

$$v_1 = \frac{2p_{i+1} + p_{i+2}}{3}$$

$$v_2 = \frac{p_{i+1} + 2p_{i+2}}{3}$$

$$v_0 = \frac{1}{6}(p_i + 4p_{i+1} + p_{i+2})$$

$$v_3 = \frac{1}{6}(p_{i+1} + 4p_{i+2} + p_{i+3})$$

From B-Splines to Bezier

- In matrix form:

$$\begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \\ \mathbf{p}_{i+3} \end{bmatrix}$$

From B-Splines to NURBS

- What are NURBS??
- Non Uniform Rational B-Splines (NURBS)
- Why rational curves?
- Polynomial-based splines can not represent commonly-used analytic shapes such as conic section, circles, ellipses, parabolas.
- Rational Splines can achieve this goal.
- NURBS are a unified representation
  - Polynomial, conic section, etc.
  - Industry standard.

NURBS

- NURBS mathematics:

$$\mathbf{c}(u) = \frac{\sum_{i=0}^{n} \mathbf{p}_i w_i B_{i,k}(u)}{\sum_{i=0}^{n} w_i B_{i,k}(u)}$$

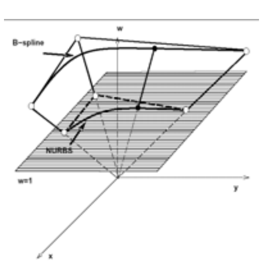
- Geometric Meaning--- obtained from projection!
- B-splines in homogenous representation

$$\text{c(u)=} \begin{bmatrix} x(u) \\ y(u) \\ z(u) \\ w(u) \end{bmatrix} = \sum_{i=0}^{n} \begin{bmatrix} \mathbf{p}_{i,x} w_i \\ \mathbf{p}_{i,y} w_i \\ \mathbf{p}_{i,z} w_i \\ w_i \end{bmatrix} B_{i,k}(u) = \sum_{i=0}^{n} \begin{bmatrix} \mathbf{p}_i w_i \\ \mathbf{w}_i \end{bmatrix} B_{i,k}(u)$$

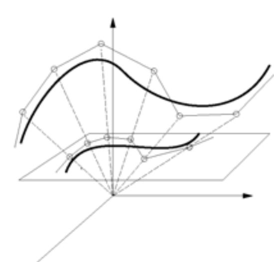
Properties of NURBS

- Represent standard shapes.
- Invariant under perspective projection.
- B-Spline is a special case.
- Weights as extra degrees of freedom.
- Can represent analytic shapes such as circles.

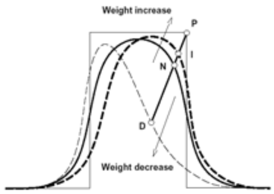
NURBS



NURBS

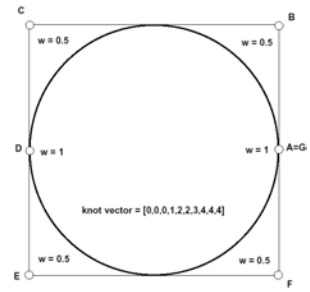


NURBS



Weight increase
P
I
N
D
Weight decrease

NURBS



C
B
w = 0.5
w = 0.5
D
w = 1
w = 1
knot vector = [0,0,0,1,2,2,3,4,4,4]
w = 0.5
w = 0.5
E
F