Solution to Homework #1

Solution to 4.6

Path diversity and lower-bounded flows

\[ \text{MIP: A/PAP (path diversity—lower-bounded flows)} \]

**constants**
\[ \delta_{edp} = 1 \text{ if link } e \text{ belongs to path } p \text{ realizing demand } d; 0, \text{ otherwise} \]
\[ h_d \quad \text{volume of demand } d \]
\[ n_d \quad \text{diversity factor for demand } d \]
\[ b_d \quad \text{lower bound on non-zero flows of demand } d \]
\[ c_e \quad \text{capacity of link } e \]

**variables**
\[ x_{dp} \quad \text{continuous flow variable allocated to path } p \text{ of demand } d \]
\[ u_{dp} \quad \text{binary variable corresponding to } x_{dp} \]

**constraints**
\[ \sum_p x_{dp} = h_d, \quad d = 1, 2, \ldots, D \]
\[ x_{dp} \leq h_d u_{dp} / n_d, \quad d = 1, 2, \ldots, D \quad p = 1, 2, \ldots, P_d \]
\[ b_d u_{dp} \leq x_{dp}, \quad d = 1, 2, \ldots, D \quad p = 1, 2, \ldots, P_d \]
\[ \sum_d \sum_p \delta_{edp} x_{dp} \leq c_e, \quad e = 1, 2, \ldots, E. \]
Solution to 4.7

Equal split among 2 or 3 paths

\textbf{MIP: A/PAP (eqsplit)}

\textbf{indices}
\begin{itemize}
  \item $d = 1, 2, \ldots, D$ demands
  \item $e = 1, 2, \ldots, E$ links
  \item $p = 1, 2, \ldots, P_d$ candidate paths for demand $d$
\end{itemize}

\textbf{constants}
\begin{itemize}
  \item $\delta_{edp}$ = 1 if link $e$ belongs to path $p$ realizing demand $d$; 0, otherwise
  \item $h_d$ volume of demand $d$
  \item $c_e$ capacity of link $e$
\end{itemize}

\textbf{variables}
\begin{itemize}
  \item $x_{dp}$ flow allocated to path $p$ of demand $d$ (continuous)
  \item $u_{dp}$ binary variable corresponding to flow allocated to path $p$ of demand $d$
\end{itemize}

\textbf{constraints}
\begin{align*}
\sum_p x_{dp} &= h_d, \quad d = 1, 2, \ldots, D \\
2 \leq \sum_p u_{dp} &\leq 3, \quad d = 1, 2, \ldots, D \\
x_{dp} &\leq \frac{(5 - \sum_p u_{dp})}{6}, \quad d = 1, 2, \ldots, D \quad p = 1, 2, \ldots, P_d \\
\sum_d \sum_p \delta_{edp} x_{dp} &\leq c_e, \quad e = 1, 2, \ldots, E.
\end{align*}

Note that the first and the fourth constraints together imply that non-zero flows of demand $d$ are equal exactly to $h_d/2$ or $h_d/3$, depending on the actual number of paths used for the demand (2 and 3, respectively).

Observe that it would be difficult to write down this problem in the node-link formulation.
Solution to 4.23

Node-link formulation of a budget constraint problem

\textbf{CVP: D/BC (budget constraint with concave dimensioning functions)} \quad \textbf{Node-Link Formulation}

\textbf{indices}
- \(d = 1, 2, \ldots, D\) demands
- \(e = 1, 2, \ldots, E\) arcs (directed links)
- \(v = 1, 2, \ldots, V\) nodes

\textbf{constants}
- \(a_{ev}\) = 1 if link \(e\) originates at node \(v\), 0 otherwise
- \(b_{ev}\) = 1 if link \(e\) terminates in node \(v\), 0 otherwise
- \(s_d\) source node of demand \(d\)
- \(t_d\) sink node of demand \(d\)
- \(h_d\) reference volume of demand \(d\)
- \(F_e(\cdot)\) concave dimensioning function of link \(e\)
- \(\xi_e\) unit cost of link \(e\)
- \(B\) given budget

\textbf{variables}
- \(x_{od}\) flow realizing demand \(d\) allocated to link \(e\) (continuous non-negative)
- \(y_e\) capacity of link \(e\) (continuous non-negative)
- \(r\) proportion of the realized demand volumes (continuous non-negative)

\textbf{objective}

\begin{align*}
\text{maximize} \quad & r \\
\text{constraints} \quad & \sum_v a_{ev} x_{od} - \sum_e b_{ev} x_{od} = \begin{cases} 
    rh_d, & \text{if } v = s_d \\
    0, & \text{if } v = t_d, \quad v = 1, 2, \ldots, V \quad d = 1, 2, \ldots, D \\
    -rh_d, & \text{if } v \neq s_d, t_d 
\end{cases} \\
& \sum_e \xi_e F_e(\sum_d x_{od}) \leq B.
\end{align*}

Observe that using the piecewise linear approximation of the dimensioning functions \(F_e(\cdot)\) would involve an additional term in the objective function and the objective function would look as follows (see solution to Exercise 4.22):

\textbf{objective}

\begin{align*}
\text{minimize} \quad & -r + \sum_v \sum_e (a_{ek} y_{ek} + b_{ek} u_{ek}).
\end{align*}