Queuing Analysis:

Time Reversibility and Burke’s Theorem

Hongwei Zhang

http://www.cs.wayne.edu/~hzhang

Acknowledgement: this lecture is partially based on the slides of Dr. Yannis A. Korilis.

Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke’s Theorem
- Queues in Tandem

Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke's Theorem
- Queues in Tandem

Time-Reversed Markov Chains

- $\{X_n: n=0,1,\ldots\}$ irreducible aperiodic Markov chain with transition probabilities P_{ij}

$$\sum_{j=0}^{\infty} P_{ij} = 1, \quad i = 0,1,\ldots$$

- Unique stationary distribution ($\pi_j > 0$) if and only if GBE holds, i.e.,

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0,1,\ldots$$

- Process in steady state:

$$\Pr\{X_n = j\} = \pi_j = \lim_{n\to\infty} \Pr\{X_n = j \mid X_0 = i\}$$

  - Starts at $n=-\infty$, that is $\{X_n: n = \ldots,-1,0,1,\ldots\}$, or
  - Choose initial state according to the stationary distribution

- *How does $\{X_n\}$ look "reversed" in time?*

Time-Reversed Markov Chains

- Define $Y_n=X_{\tau-n}$, for arbitrary $\tau>0$

  => $\{Y_n\}$ is the reversed process.

- <u>Proposition 1:</u>
  - $\{Y_n\}$ is a Markov chain with transition probabilities:

$$P_{ij}^{*} = \frac{\pi_j P_{ji}}{\pi_i}, \quad i, j = 0, 1, \ldots$$

  - $\{Y_n\}$ has the same stationary distribution π_j with the forward chain $\{X_n\}$
    - The reversed chain corresponds to the same process, looked at in the reversed-time direction

Time-Reversed Markov Chains

Proof of Proposition 1:

$$P^*_{ij} = P\{Y_m = j \mid Y_{m-1} = i, Y_{m-2} = i_2, \ldots, Y_{m-k} = i_k\}$$
$$= P\{X_{\tau-m} = j \mid X_{\tau-m+1} = i, X_{\tau-m+2} = i_2, \ldots, X_{\tau-m+k} = i_k\}$$
$$= P\{X_n = j \mid X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k\}$$
$$= \frac{P\{X_n = j, X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k\}}{P\{X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k\}}$$
$$= \frac{P\{X_{n+2} = i_2, \ldots, X_{n+k} = i_k \mid X_n = j, X_{n+1} = i\} P\{X_n = j, X_{n+1} = i\}}{P\{X_{n+2} = i_2, \ldots, X_{n+k} = i_k \mid X_{n+1} = i\} P\{X_{n+1} = i\}}$$
$$= \frac{P\{X_n = j, X_{n+1} = i\}}{P\{X_{n+1} = i\}} = P\{X_n = j \mid X_{n+1} = i\} = P\{Y_m = j \mid Y_{m-1} = i\}$$
$$= \frac{P\{X_{n+1} = i \mid X_n = j\} P\{X_n = j\}}{P\{X_{n+1} = i\}} = \frac{P_{ji}\pi_j}{\pi_i}$$

$$\Rightarrow \sum_{i=0}^{\infty} \pi_i P^*_{ij} = \sum_{i=0}^{\infty} \pi_i \frac{\pi_j P_{ji}}{\pi_i} = \pi_j \sum_{i=0}^{\infty} P_{ji} = \pi_j$$

Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke's Theorem
- Queues in Tandem

Reversibility

- Stochastic process $\{X(t)\}$ is called *reversible* if $(X(t_1), X(t_2),\ldots, X(t_n))$ and $(X(\tau-t_1), X(\tau-t_2),\ldots, X(\tau-t_n))$ have the same probability distribution, for all $\tau, t_1,\ldots, t_n$

- Markov chain $\{X_n\}$ is *reversible* if and only if the transition probabilities of forward and reversed chains are equal, i.e.,

$$P_{ij} = P_{ij}^{*}$$

- Detailed Balance Equations ↔ Reversibility

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad i, j = 0, 1, \ldots$$

Reversibility – Discrete-Time Chains

- <u>Theorem 1:</u> If there exists a set of positive numbers $\{\pi_j\}$, that sum up to 1 and satisfy:

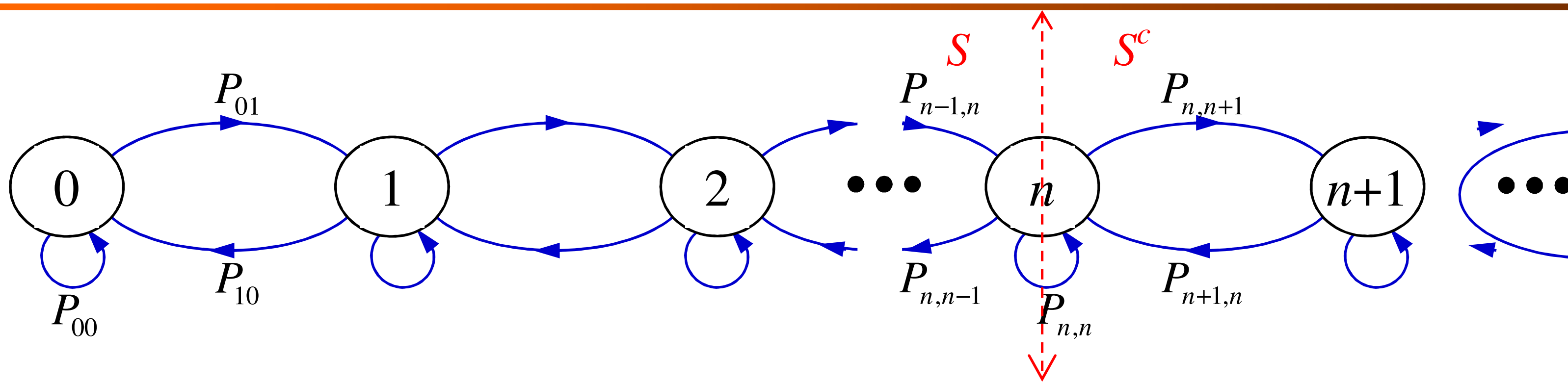
$$\pi_i P_{ij} = \pi_j P_{ji}, \quad i, j = 0, 1, \ldots$$

  Then:

  1. $\{\pi_j\}$ is the unique stationary distribution
  2. The Markov chain is reversible

- <u>Example:</u> Discrete-time birth-death processes are reversible, since they satisfy the DBE

Example: Birth-Death Process



- One-dimensional Markov chain with transitions only between neighboring states: $P_{ij}=0$, if $|i-j|>1$
- Detailed Balance Equations (DBE)

$$\pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n} \quad n = 0,1,\ldots$$

- Proof: GBE with $S=\{0,1,\ldots,n\}$ give:

$$\sum_{j=0}^{n}\sum_{i=n+1}^{\infty} \pi_j P_{ji} = \sum_{j=0}^{n}\sum_{i=n+1}^{\infty} \pi_i P_{ij} \Rightarrow \pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n}$$

Time-Reversed Markov Chains (Revisited)

- Theorem 2: Irreducible Markov chain with transition probabilities P_{ij}. If there exist:
  - A set of transition probabilities P_{ij}^*, **with $\sum_j P_{ij}^* = 1$**, $i \geq 0$, and
  - A set of positive numbers $\{\pi_j\}$, that sum up to 1, such that

$$\pi_i P_{ij}^* = \pi_j P_{ji}, \quad i, j \geq 0$$

  Then:
  - P_{ij}^* are the transition probabilities of the reversed chain, and
  - $\{\pi_j\}$ is the stationary distribution of the forward and the reversed chains

- Remark: Used to find the stationary distribution, by guessing the transition probabilities of the reversed chain – even if the process is not reversible

Continuous-Time Markov Chains

- {X(t): -∞< t <∞} irreducible aperiodic Markov chain with transition rates q_{ij}, $i \neq j$
- Unique stationary distribution ($p_i > 0$) if and only if:

$$p_j \sum_{i \neq j} q_{ji} = \sum_{i \neq j} p_i q_{ij}, \quad j = 0,1,\ldots$$

- Process in steady state – e.g., started at $t = -\infty$:

$$\Pr\{X(t) = j\} = p_j = \lim_{t \to \infty} \Pr\{X(t) = j \mid X(0) = i\}$$

- If $\{\pi_j\}$ is the stationary distribution of the embedded discrete-time chain:

$$p_j = \frac{\pi_j / \nu_j}{\sum_i \pi_i / \nu_i}, \quad \nu_j \equiv \sum_{i \neq j} q_{ji}, \quad j = 0,1,\ldots$$

Reversed Continuous-Time Markov Chains

- Reversed chain {Y(t)}, with Y(t)=X(τ-t), for arbitrary τ>0
- <u>Proposition 2:</u>

1. {Y(t)} is a continuous-time Markov chain with transition rates:

$$q_{ij}^{*} = \frac{p_j q_{ji}}{p_i}, \quad i, j = 0,1,\ldots, i \neq j$$

2. {Y(t)} has the same stationary distribution $\{p_j\}$ with the forward chain

- <u>Remark:</u> The transition rate out of state *i* in the reversed chain is equal to the transition rate out of state *i* in the forward chain

$$\sum_{j \neq i} q_{ij}^{*} = \frac{\sum_{j \neq i} p_j q_{ji}}{p_i} = \frac{p_i \sum_{j \neq i} q_{ij}}{p_i} = \sum_{j \neq i} q_{ij} = \nu_i, \quad i = 0,1,\ldots$$

Reversibility – Continuous-Time Chains

- Markov chain {X(t)} is *reversible* if and only if the transition rates of forward and reversed chains are equal $q_{ij} = q^*_{ij}$, or equivalently

$$p_i q_{ij} = p_j q_{ji}, \quad i, j = 0, 1, \ldots, i \neq j$$

  i.e., Detailed Balance Equations ↔ Reversibility

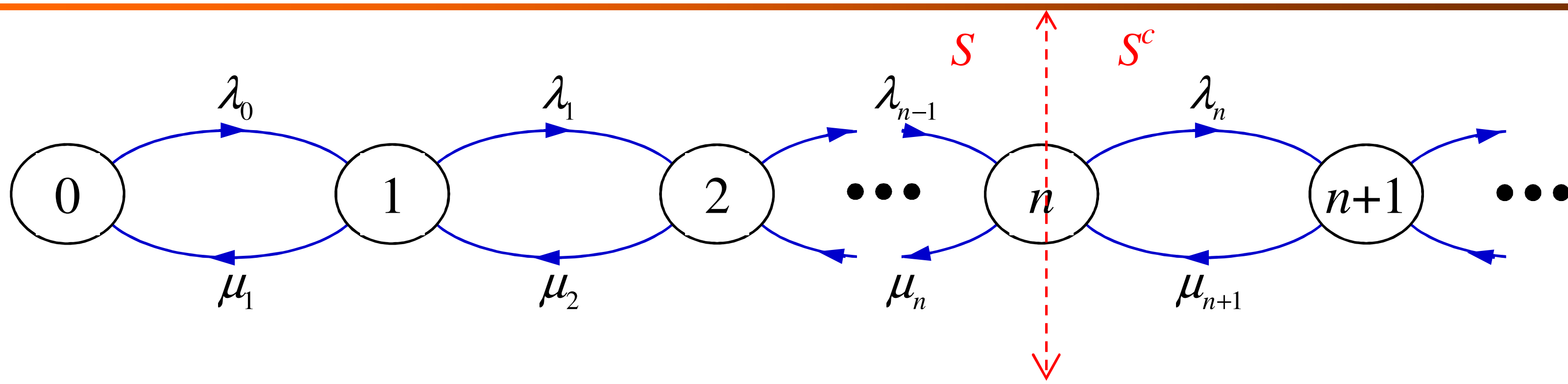
- Theorem 3: If there exists a set of positive numbers $\{p_j\}$, that sum up to 1 and satisfy:

$$p_i q_{ij} = p_j q_{ji}, \quad i, j = 0, 1, \ldots, i \neq j$$

  Then:

  1. $\{p_j\}$ is the unique stationary distribution
  2. The Markov chain is reversible

Example: Birth-Death Process



- Transitions only between neighboring states

$$q_{i,i+1} = \lambda_i, \quad q_{i,i-1} = \mu_i, \quad q_{ij} = 0, \ |i - j| > 1$$

- Detailed Balance Equations

$$\lambda_n p_n = \mu_{n+1} p_{n+1}, \quad n = 0,1,\ldots$$

- Proof: GBE with $S = \{0,1,\ldots,n\}$ give:

$$\sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_j q_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_i q_{ij} \Rightarrow \lambda_n p_n = \mu_{n+1} p_{n+1}$$

- M/M/1, M/M/c, M/M/∞

Reversed Continuous-Time Markov Chains (Revisited)

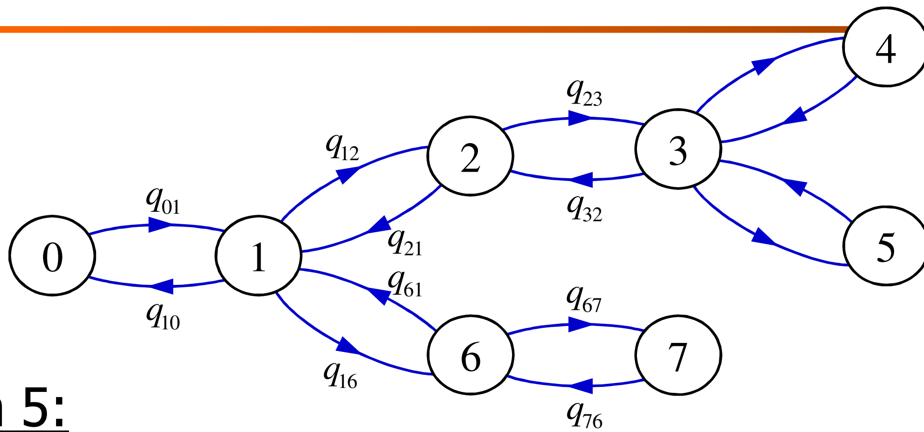
- Theorem 4: Irreducible continuous-time Markov chain with transition rates q_{ij}. If there exist:
  - A set of transition rates q_{ij}^*, with $\sum_{j\neq i} q_{ij}^* = \sum_{j\neq i} q_{ij}$, $i \geq 0$, and
  - A set of positive numbers $\{p_j\}$, that sum up to 1, such that

$$p_i q_{ij}^* = p_j q_{ji}, \quad i, j \geq 0, i \neq j$$

  Then:
  - q_{ij}^* are the transition rates of the reversed chain, and
  - $\{p_j\}$ is the stationary distribution of the forward and the reversed chains

- Remark: Used to find the stationary distribution, by guessing the transition probabilities of the reversed chain – even if the process is not reversible

Reversibility: Trees



Theorem 5:

- Irreducible Markov chain, with transition rates that satisfy $q_{ij}>0 \leftrightarrow q_{ji}>0$
- Form a graph for the chain, where states are the nodes, and for each $q_{ij}>0$, there is a directed arc $i \rightarrow j$

Then, if graph is a tree – contains no loops – then Markov chain is reversible

Remarks:

- Sufficient condition for reversibility
- Generalization of one-dimensional birth-death process

Kolmogorov's Criterion (Discrete Chain)

- Detailed balance equations determine whether a Markov chain is reversible or not, based on stationary distribution and *transition probabilities*
- Should be able to derive a reversibility criterion based only on the transition probabilities!

- Theorem 6: A discrete-time Markov chain is reversible *if and only if*:

$$P_{i_1 i_2} P_{i_2 i_3} \cdots P_{i_{n-1} i_n} P_{i_n i_1} = P_{i_1 i_n} P_{i_n i_{n-1}} \cdots P_{i_3 i_2} P_{i_2 i_1}$$

for any finite sequence of states: i_1, i_2,..., i_n, and any n

- Intuition: Probability of traversing any loop $i_1 \to i_2 \to \ldots \to i_n \to i_1$ is equal to the probability of traversing the same loop in the reverse direction $i_1 \to i_n \to \ldots \to i_2 \to i_1$

Kolmogorov's Criterion (Continuous Chain)

- Detailed balance equations determine whether a Markov chain is reversible or not, based on stationary distribution and *transition rates*
- Should be able to derive a reversibility criterion based only on the transition rates!

- Theorem 7: A continuous-time Markov chain is reversible *if and only if*:

$$q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i_1 i_n} q_{i_n i_{n-1}} \cdots q_{i_3 i_2} q_{i_2 i_1}$$

  for any finite sequence of states: i_1, i_2,..., i_n, and any n

- Intuition: Product of transition rates along any loop $i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_n \rightarrow i_1$ is equal to the product of transition rates along the same loop traversed in the reverse direction $i_1 \rightarrow i_n \rightarrow \ldots \rightarrow i_2 \rightarrow i_1$

Kolmogorov's Criterion (proof)

Proof of Theorem 6:

- Necessary: If the chain is reversible the DBE hold

$$\left.\begin{aligned} \pi_1 P_{i_1 i_2} &= \pi_2 P_{i_2 i_1} \\ \pi_2 P_{i_2 i_3} &= \pi_3 P_{i_3 i_2} \\ &\vdots \\ \pi_{n-1} P_{i_{n-1} i_n} &= \pi_n P_{i_n i_{n-1}} \\ \pi_n P_{i_n i_1} &= \pi_1 P_{i_1 i_n} \end{aligned}\right\} \Rightarrow P_{i_1 i_2} P_{i_2 i_3} \cdots P_{i_{n-1} i_n} P_{i_n i_1} = P_{i_1 i_n} P_{i_n i_{n-1}} \cdots P_{i_3 i_2} P_{i_2 i_1}$$

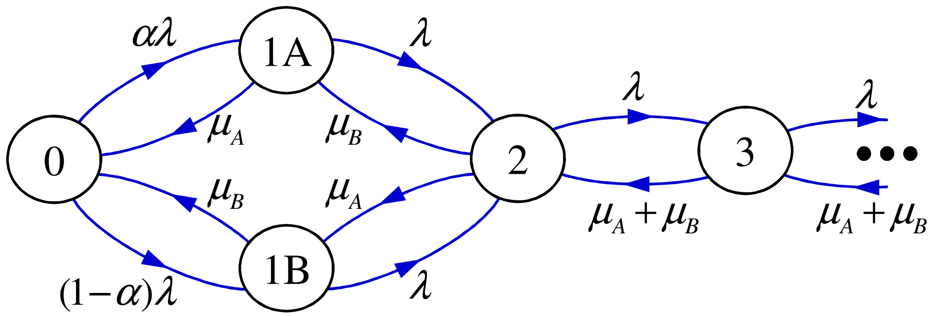
- Sufficient: Fixing two states $i_1=i$, and $i_n=j$ and summing over all states $i_2,\ldots,i_{n-1}$ we have

$$P_{i,i_2} P_{i_2 i_3} \cdots P_{i_{n-1},j} P_{ji} = P_{ij} P_{j,i_{n-1}} \cdots P_{i_3 i_2} P_{i_2,i} \Rightarrow P_{ij}^{n-1} P_{ji} = P_{ij} P_{ji}^{n-1}$$

Taking the limit $n \to \infty$

$$\lim_{n\to\infty} P_{ij}^{n-1} \cdot P_{ji} = P_{ij} \cdot \lim_{n\to\infty} P_{ji}^{n-1} \Rightarrow \pi_j P_{ji} = P_{ij} \pi_i$$

Example: M/M/2 Queue with Heterogeneous Servers



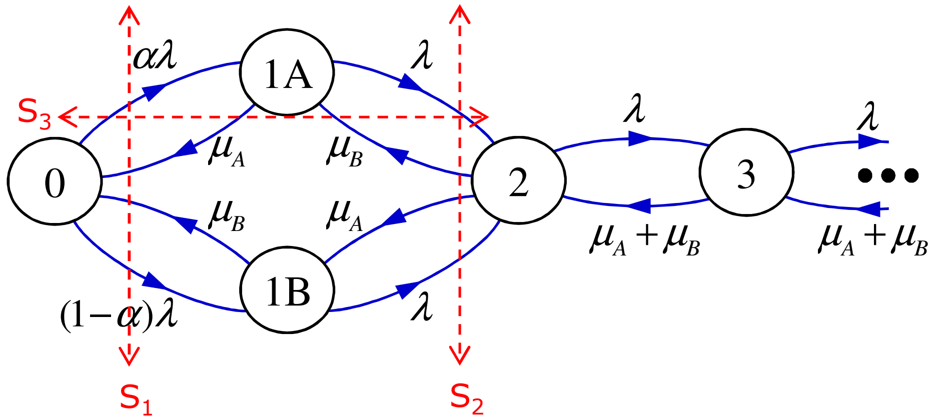
- M/M/2 queue. Servers A and B with service rates μ_A and μ_B respectively. When the system empty, arrivals go to A with probability α and to B with probability 1-α. Otherwise, the head of the queue takes the first free server.
- Need to keep track of which server is busy when there is 1 customer in the system. Denote the two possible states by: 1A and 1B.
- Reversibility: we only need to check the loop 0→1A→2→1B→0:

$$q_{0,1A}q_{1A,2}q_{2,1B}q_{1B,0} = \alpha\lambda \cdot \lambda \cdot \mu_A \cdot \mu_B \qquad q_{0,1B}q_{1B,2}q_{2,1A}q_{1A,0} = (1-\alpha)\lambda \cdot \lambda \cdot \mu_B \cdot \mu_A$$

  - Reversible if and only if α=1/2.

What happens when $\mu_A=\mu_B$, and α≠1/2?

Example: M/M/2 Queue with Heterogeneous Servers



$$p_n = p_2 \left(\frac{\lambda}{\mu_A + \mu_B} \right)^{n-2}, \quad n = 2, 3, \ldots$$

$$\left.\begin{aligned} \lambda p_0 &= \mu_A p_{1A} + \mu_B p_{1B} \\ (\mu_A + \mu_B) p_2 &= \lambda (p_{1A} + p_{1B}) \\ (\mu_A + \lambda) p_{1A} &= \alpha\lambda p_0 + \mu_B p_2 \end{aligned}\right\} \Rightarrow \begin{aligned} p_{1A} &= p_0 \frac{\lambda}{\mu_A} \frac{\lambda + \alpha(\mu_A + \mu_B)}{2\lambda + \mu_A + \mu_B} \\ p_{1B} &= p_0 \frac{\lambda}{\mu_B} \frac{\lambda + (1-\alpha)(\mu_A + \mu_B)}{2\lambda + \mu_A + \mu_B} \\ p_2 &= p_0 \frac{\lambda^2}{\mu_A \mu_B} \frac{\lambda + (1-\alpha)\mu_A + \alpha\mu_B}{2\lambda + \mu_A + \mu_B} \end{aligned}$$

$$p_0 + p_{1A} + p_{1B} + \sum_{n=2}^{\infty} p_n = 1 \Rightarrow p_0 = \left[1 + \frac{\lambda}{\mu_A + \mu_B - \lambda} \frac{\lambda^2}{\mu_A \mu_B} \frac{\lambda + (1-\alpha)\mu_A + \alpha\mu_B}{2\lambda + \mu_A + \mu_B} \right]^{-1}$$

Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke's Theorem
- Queues in Tandem

Multidimensional Markov Chains

Theorem 8:

- $\{X_1(t)\}$, $\{X_2(t)\}$: *independent* Markov chains
- $\{X_i(t)\}$: reversible
- $\{X(t)\}$, with $X(t)=(X_1(t), X_2(t))$: vector-valued stochastic process
- ➡ $\{X(t)\}$ is a Markov chain
- ➡ $\{X(t)\}$ is reversible

Multidimensional Chains:

- Queueing system with two classes of customers, each having its own stochastic properties – track the number of customers from each class
- Study the "joint" evolution of two queueing systems – track the number of customers in each system

Example: Two Independent M/M/1 Queues

- Two independent M/M/1 queues. The arrival and service rates at queue i are λ_i and μ_i respectively. Assume $\rho_i = \lambda_i/\mu_i < 1$.
- $\{(N_1(t), N_2(t))\}$ is a Markov chain.
- Probability of n_1 customers at queue 1, and n_2 at queue 2, at steady-state

$$p(n_1, n_2) = (1-\rho_1)\rho_1^{n_1} \cdot (1-\rho_2)\rho_2^{n_2} = p_1(n_1) \cdot p_2(n_2)$$

- "Product-form" distribution
- Generalizes for any number K of independent queues, M/M/1, M/M/c, or M/M/∞. If $p_i(n_i)$ is the stationary distribution of queue i:

$$p(n_1, n_2, \ldots, n_K) = p_1(n_1) p_2(n_2) \ldots p_K(n_K)$$

Example (contd.)

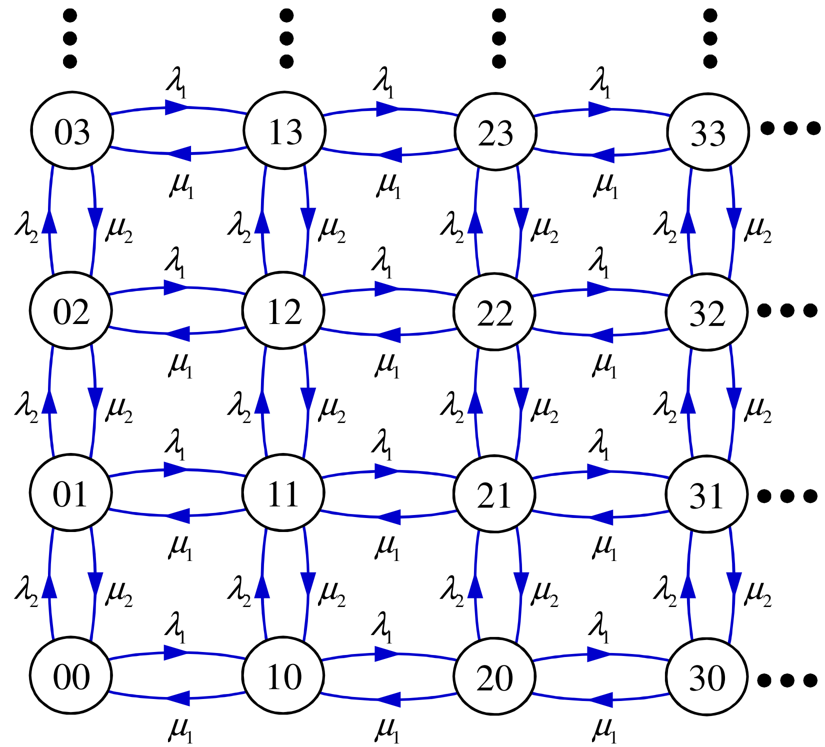
- Stationary distribution:

$$p(n_1, n_2) = \left(1 - \frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \left(1 - \frac{\lambda_2}{\mu_2}\right)\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}$$

- Detailed Balance Equations:

$$\mu_1 p(n_1 + 1, n_2) = \lambda_1 p(n_1, n_2)$$
$$\mu_2 p(n_1, n_2 + 1) = \lambda_2 p(n_1, n_2)$$

- Verify that the Markov chain is reversible – Kolmogorov criterion



Truncation of a Reversible Markov Chain

- Theorem 9: {X(t)} reversible Markov process with state space *S*, and stationary distribution $\{p_j: j \in S\}$. Truncated to a set $E \subset S$, such that the resulting chain {Y(t)} is *irreducible*. Then, {Y(t)} is reversible and has stationary distribution:

$$\tilde{p}_j = \frac{p_j}{\sum_{k \in E} p_k}, \quad j \in E$$

- Remark: This is the conditional probability that, in steady-state, the original process is at state *j*, given that it is somewhere in *E*

- Proof: Verify that:

$$\tilde{p}_j q_{ji} = \tilde{p}_i q_{ij} \Leftrightarrow \frac{p_j}{\sum_{k \in E} p_k} q_{ji} = \frac{p_i}{\sum_{k \in E} p_k} q_{ij} \Leftrightarrow p_j q_{ji} = p_i q_{ij}, \quad i, j \in S; i \neq j$$

$$\sum_{j \in E} \tilde{p}_j = \sum_{j \in E} \frac{p_j}{\sum_{k \in E} p_k} = 1$$

Example: Two Queues with Joint Buffer

- The two independent M/M/1 queues of the previous example share a common buffer of size B – arrival that finds B customers *waiting* is blocked
- State space restricted to

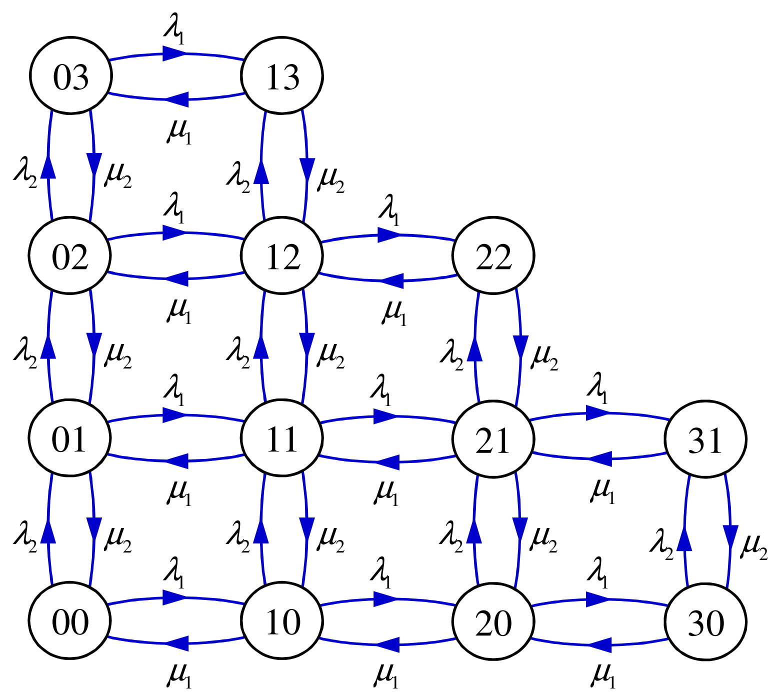
$$E = \{(n_1, n_2) : (n_1 - 1)^+ + (n_2 - 1)^+ \le B\}$$

- Distribution of truncated chain:

$$p(n_1, n_2) = p(0,0) \cdot \rho_1^{n_1} \rho_2^{n_2}, \ (n_1, n_2) \in E$$

- Normalizing:

$$p(0,0) = \left[\sum_{(n_1, n_2) \in E} \rho_1^{n_1} \rho_2^{n_2} \right]^{-1}$$



State diagram for $B = 2$

- Theorem specifies joint distribution up to the normalization constant
- Calculation of normalization constant is often tedious

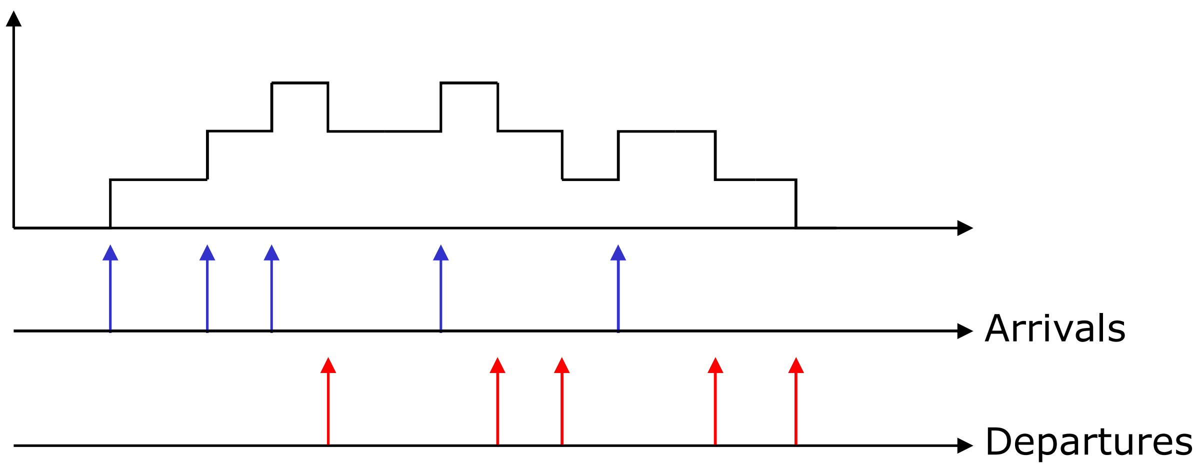
Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke’s Theorem
- Queues in Tandem

Birth-death process

- {X(t)} birth-death process with stationary distribution $\{p_j\}$
- Arrival epochs: points of increase for {X(t)}

  Departure epoch: points of decrease for {X(t)}
- {X(t)} completely determines the corresponding arrival and departure processes

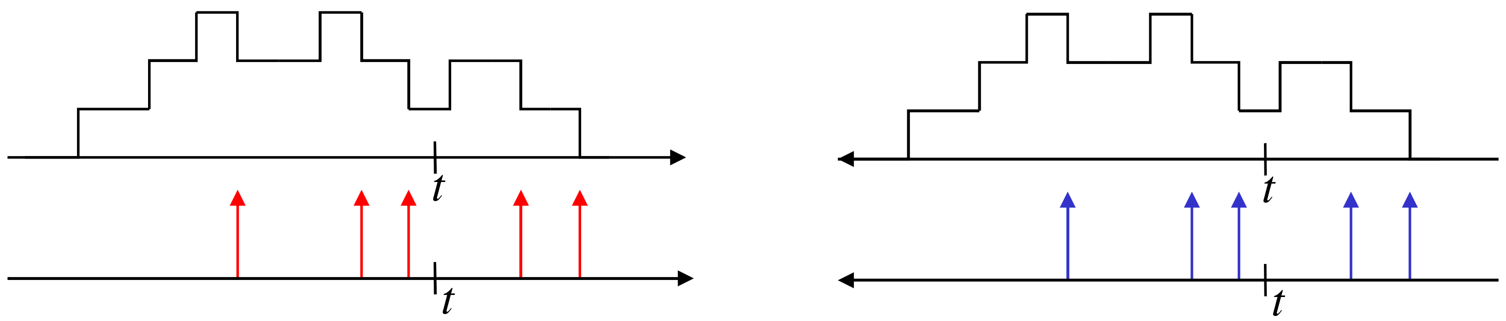


Forward & reversed chains of M/M/* queues

- Poisson arrival process: $\lambda_j=\lambda$, for all j
  - Birth-death process called a (λ, μ_j)-process
  - Examples: M/M/1, M/M/c, M/M/∞ queues
- Poisson arrivals → LAA: for any time t, future arrivals are independent of $\{X(s): s \leq t\}$

- (λ, μ_j)-process at steady state is reversible: forward and reversed chains are stochastically identical
- => Arrival processes of the forward and reversed chains are stochastically identical
  - => Arrival process of the reversed chain is Poisson with rate λ
  - + "the arrival epochs of the reversed chain are the departure epochs of the forward chain" => **Departure process of the forward chain is Poisson with rate λ**

Forward & reversed chains of M/M/* queues (contd.)



- Reversed chain: arrivals after time t are independent of the chain history up to time t (LAA)
- => Forward chain: departures prior to time t and future of the chain {X(s): s≥t} are independent

Burke's Theorem

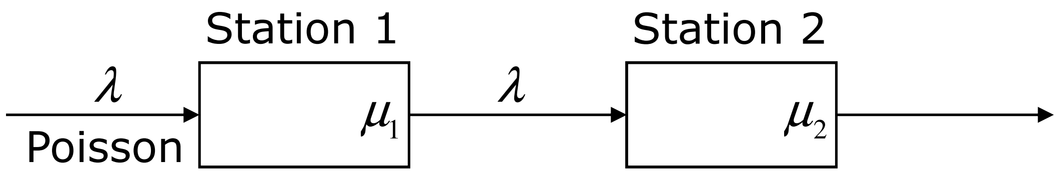
- Theorem 10: Consider an M/M/1, M/M/c, or M/M/∞ system with arrival rate λ. *Suppose that the system starts at steady-state*. Then:
  1. The departure process is Poisson with rate λ
  2. At each time *t*, the number of customers in the system is independent of the departure times prior to *t*

- Fundamental result for study of networks of M/M/* queues, where output process from one queue is the input process of another

Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke's Theorem
- Queues in Tandem

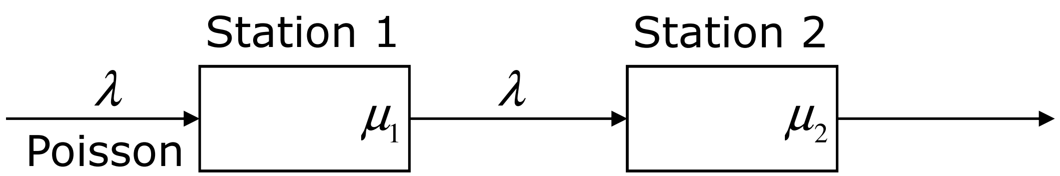
Single-Server Queues in Tandem



- Customers arrive at queue 1 according to Poisson process with rate λ.
- Service times exponential with mean $1/\mu_i$. *Assume service times of a customer in the two queues are independent*.
- Assume $\rho_i=\lambda/\mu_i<1$
- What is the joint *stationary* distribution of N_1 and N_2 – number of customers in each queue?
- Result: in *steady state* the queues are independent and

$$p(n_1,n_2)=(1-\rho_1)\rho_1^{n_1}\cdot(1-\rho_2)\rho_2^{n_2}=p_1(n_1)\cdot p_2(n_2)$$

Single-Server Queues in Tandem



- Q1 is a M/M/1 queue. At steady state its departure process is Poisson with rate λ. Thus Q2 is also M/M/1.
- Marginal stationary distributions:

$$p_1(n_1) = (1-\rho_1)\rho_1^{n_1}, \quad n_1 = 0,1,\ldots \qquad p_2(n_2) = (1-\rho_2)\rho_2^{n_2}, \quad n_2 = 0,1,\ldots$$

- To complete the proof: establish independence at steady state
- Q1 at steady state: at time t, $N_1(t)$ is independent of departures prior to t, which are arrivals at Q2 up to t. Thus $N_1(t)$ and $N_2(t)$ independent:

$$P\{N_1(t)=n_1, N_2(t)=n_2\} = P\{N_1(t)=n_1\}P\{N_2(t)=n_2\} = p_1(n_1)\cdot P\{N_2(t)=n_2\}$$

- Letting t→∞, the joint stationary distribution

$$p(n_1,n_2) = p_1(n_1)\cdot p_2(n_2) = (1-\rho_1)\rho_1^{n_1}\cdot(1-\rho_2)\rho_2^{n_2}$$

Queues in Tandem

- Theorem 11: Network consisting of *K* single-server queues in tandem. Service times at queue *i* exponential with rate μ_i, independent of service times at any queue $j \neq i$. Arrivals at the first queue are Poisson with rate λ. The stationary distribution of the network is:

$$p(n_1,\ldots,n_K)=\prod_{i=1}^{K}(1-\rho_i)\rho_i^{n_i}, \quad n_i=0,1,\ldots; i=1,\ldots,K$$

- At *steady state* the queues are independent; the distribution of queue *i* is that of an isolated M/M/1 queue with arrival and service rates λ and μ_i

$$p_i(n_i)=(1-\rho_i)\rho_i^{n_i}, \quad n_i=0,1,\ldots$$

Are the queues independent if not in steady state? Are stochastic processes $\{N_1(t)\}$ and $\{N_2(t)\}$ independent?

Queues in Tandem: State-Dependent Service Rates

- Theorem 12: Network consisting of K queues in tandem. Service times at queue i exponential with rate $\mu_i(n_i)$ when there are n_i customers in the queue – independent of service times at any queue $j \neq i$. Arrivals at the first queue are Poisson with rate λ. The stationary distribution of the network is:

$$p(n_1,\ldots,n_K) = \prod_{i=1}^{K} p_i(n_i), \quad n_i = 0,1,\ldots; i = 1,\ldots,K$$

  where $\{p_i(n_i)\}$ is the stationary distribution of queue i in isolation with Poisson arrivals with rate λ

- Examples: ./M/c and ./M/∞ queues
  - If queue i is ./M/∞, then:

$$p_i(n_i) = \frac{(\lambda/\mu_i)^{n_i}}{n_i!} e^{-\lambda/\mu_i}, \quad n_i = 0,1,\ldots$$

Summary

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
  - Multi-dimensional Markov chains
- Burke's Theorem
- Queues in Tandem