Queuing Analysis:

Review of Markov Chain Theory

Hongwei Zhang

http://www.cs.wayne.edu/~hzhang

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Outline

- Markov Chain
- Discrete-Time Markov Chains
- Calculating Stationary Distribution
- Global Balance Equations
- Birth-Death Process
  - Detailed Balance Equations
- Generalized Markov Chains
- Continuous-Time Markov Chains

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Markov Chain?

- Stochastic process that takes values in a *countable* set
  - Example: {0,1,2,…,m}, or {0,1,2,…}
  - Elements represent possible "states"
  - Chain transits from state to state
- *Memoryless (Markov) Property*: Given the present state, future transitions of the chain are independent of past history
- Markov Chains: discrete- or continuous- time

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Discrete-Time Markov Chain

- Discrete-time stochastic process $\{X_n: n = 0,1,2,\ldots\}$
- Takes values in $\{0,1,2,\ldots\}$
- Memoryless property:

$$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = P\{X_{n+1} = j \mid X_n = i\}$$
$$P_{ij} = P\{X_{n+1} = j \mid X_n = i\}$$

- Transition probabilities P_{ij}

$$P_{ij} \geq 0, \qquad \sum_{j=0}^{\infty} P_{ij} = 1$$

- Transition probability matrix $P=[P_{ij}]$

Chapman-Kolmogorov Equations

- n step transition probabilities

$$P_{ij}^{n} = P\{X_{n+m} = j \mid X_{m} = i\}, \quad n,m \geq 0,\ i,j \geq 0$$

- How to calculate?
  - Chapman-Kolmogorov equations

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}, \quad n,m \geq 0,\ i,j \geq 0$$

    - P_{ij}^{n} is element (i, j) in matrix P^n
    - Recursive computation of *state* probabilities

State Probabilities – Stationary Distribution

- State probabilities (time-dependent)

$$\pi_j^n = P\{X_n = j\}, \qquad \pi^n = (\pi_0^n, \pi_1^n, \ldots)$$

$$P\{X_n = j\} = \sum_{i=0}^{\infty} P\{X_{n-1} = i\} P\{X_n = j \mid X_{n-1} = i\} \Rightarrow \pi_j^n = \sum_{i=0}^{\infty} \pi_i^{n-1} P_{ij}$$

In matrix form:

$$\pi^n = \pi^{n-1} P = \pi^{n-2} P^2 = \ldots = \pi^0 P^n$$

- If time-dependent distribution converges to a limit

$$\pi = \lim_{n \to \infty} \pi^n \qquad \pi = \pi P$$

π is called the *stationary distribution* (or *steady state distribution*)

  - existence depends on the structure of Markov chain

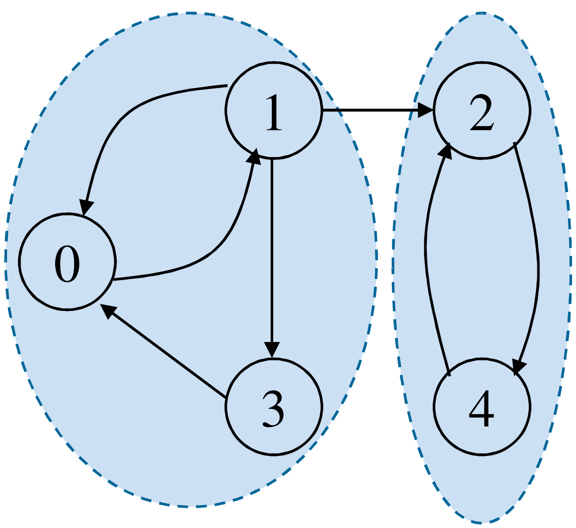
Classification of Markov Chains

Irreducible:

- States i and j communicate:

$$\exists n, m: \ P_{ij}^{n} > 0, P_{ji}^{m} > 0$$

- Irreducible Markov chain: all states communicate

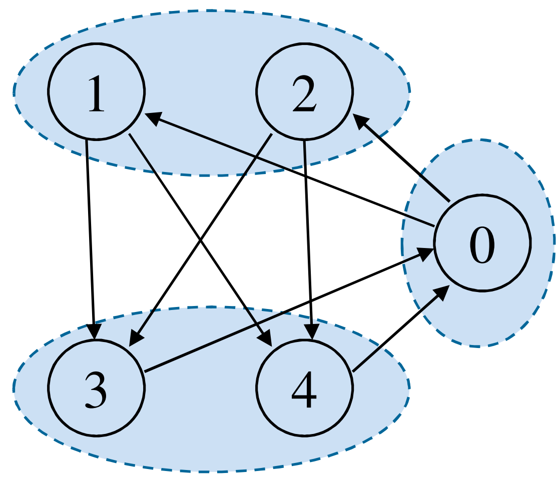


Aperiodic:

- State i is periodic:

$$\exists\, d > 1: \ P_{ii}^{n} > 0 \Rightarrow n = \alpha d$$

- Aperiodic Markov chain: none of the states is periodic



Limit Theorems

<u>Theorem 1:</u> Irreducible aperiodic Markov chain

- For every state j, the following limit

$$\pi_j = \lim_{n\to\infty} P\{X_n = j \mid X_0 = i\}, \quad i = 0,1,2,\ldots$$

  exists and is independent of initial state i

- $N_j(k)$: number of visits to state j up to time k

$$P\left\{\pi_j = \lim_{k\to\infty} \frac{N_j(k)}{k} \mid X_0 = i\right\} = 1$$

=>π_j: *frequency the process visits state j*

Existence of Stationary Distribution

<u>Theorem 2:</u> Irreducible aperiodic Markov chain. There are two possibilities for scalars:

$$\pi_j = \lim_{n\to\infty} P\{X_n = j \mid X_0 = i\} = \lim_{n\to\infty} P_{ij}^n$$

1. $\pi_j = 0$, for all states j ➡ No stationary distribution
2. $\pi_j > 0$, for all states j ➡ π is the *unique* stationary distribution

<u>Remark:</u> If the number of states is finite, case 2 is the only possibility

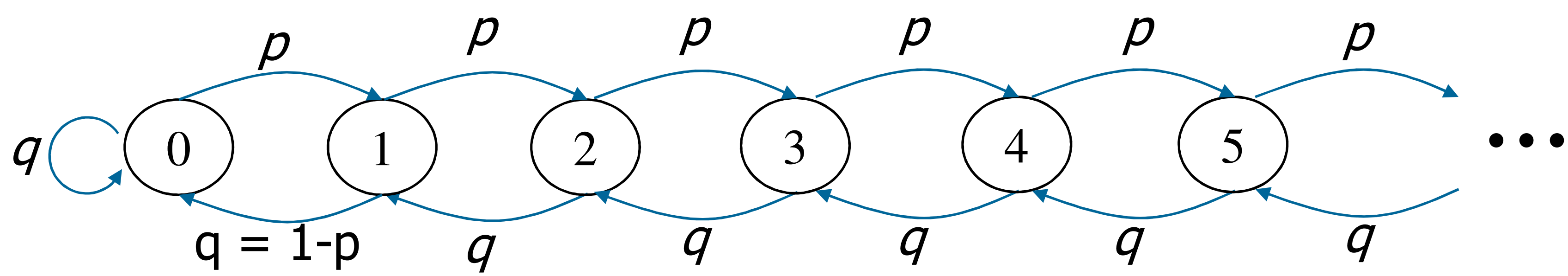
Ergodic Markov Chains

- A state j is *positive recurrent* if the process returns to state j "infinitely often"
- Formal definition:
  - $F_{ij}(n)$ ($n \geq 1$): the probability, given $X_0 = i$, that state j occurs at some time between 1 and n inclusive
  - T_{ij}: the first passage time from i to j
  - A state j is *recurrent* (or *persistent*) if $F_{jj}(\infty) = 1$, and *transient* otherwise
  - A state j is *positive recurrent* (or *non-null persistent*) if $F_{jj}(\infty) = 1$ and $E(T_{jj}) < \infty$
  - A state j is *null recurrent* (or *null persistent*) if $F_{jj}(\infty) = 1$ but $E(T_{jj}) = \infty$

  - Note: "positive recurrent => irreducible" always hold, but "irreducible => positive recurrent" is guaranteed to hold only for finite MC

Ergodic MC (contd.)

- Example: a MC with countably infinite state space



  - All states are positive recurrent if $p < ½$, null recurrent if $p = ½$, and transient if $p > ½$
- A state is *ergodic* if it is aperiodic and positive recurrent
- A MC is ergodic if every state is ergodic
- Ergodic chains have a unique stationary distribution

$$\pi_j = 1/E(T_{jj}),\ j = 0, 1, 2, \ldots$$

  - Note: Ergodicity ⇒ Time Averages = Stochastic Averages

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Calculation of Stationary Distribution

A. Finite number of states

- Solve explicitly the system of equations

$$\pi_j = \sum_{i=0}^{m} \pi_i P_{ij}, \quad j = 0, 1, \ldots, m$$

$$\sum_{i=0}^{m} \pi_i = 1$$

- Or, numerically from P^n which converges to a matrix with rows equal to π
  - Suitable for a small number of states

B. Infinite number of states

- Cannot apply previous methods to problem of infinite dimension
- Guess a solution to recurrence:

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots,$$

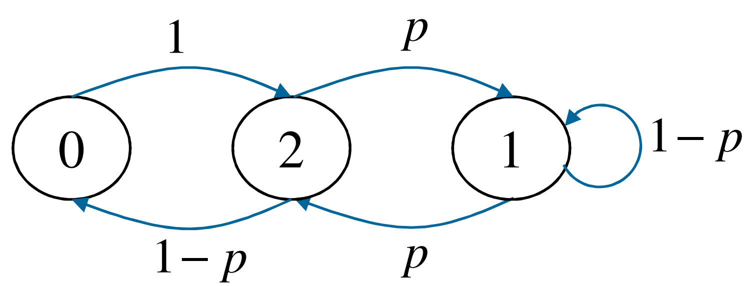
$$\sum_{i=0}^{\infty} \pi_i = 1$$

Example: Finite Markov Chain

- Absent-minded professor uses two umbrellas when commuting between home and office.
- If it rains and an umbrella is available at her location, she takes it. If it does not rain, she always forgets to take an umbrella.
- Let p be the probability of rain each time she commutes.

Q: What is the probability that she gets wet on any given day?

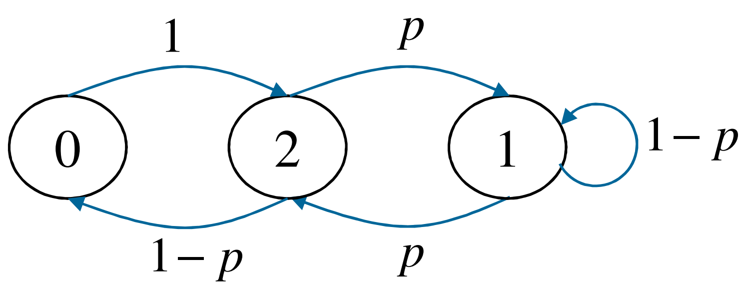
- Markov chain formulation
- i is the number of umbrellas available at her current location



- Transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

Example: Finite Markov Chain



$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

$$\begin{cases} \pi = \pi P \\ \sum_i \pi_i = 1 \end{cases} \Leftrightarrow \begin{cases} \pi_0 = (1-p)\pi_2 \\ \pi_1 = (1-p)\pi_1 + p\pi_2 \\ \pi_2 = \pi_0 + p\pi_1 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \Leftrightarrow \pi_0 = \frac{1-p}{3-p},\ \pi_1 = \frac{1}{3-p},\ \pi_2 = \frac{1}{3-p}$$

$$P\{\text{gets wet}\} = \pi_0 p = p\frac{1-p}{3-p}$$

Example: Finite Markov Chain

- Taking p = 0.1:

$$\pi = \left(\frac{1-p}{3-p}, \frac{1}{3-p}, \frac{1}{3-p} \right) = (0.310, 0.345, 0.345)$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0.9 & 0.1 \\ 0.9 & 0.1 & 0 \end{bmatrix}$$

- Numerically determine limit of P^n

$$\lim_{n \to \infty} P^n = \begin{bmatrix} 0.310 & 0.345 & 0.345 \\ 0.310 & 0.345 & 0.345 \\ 0.310 & 0.345 & 0.345 \end{bmatrix} \quad (n \approx 150)$$

- Effectiveness depends on structure of P

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Global Balance Equations

- Global Balance Equations (GBE)

$$\pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \Leftrightarrow \pi_j \sum_{i \neq j} P_{ji} = \sum_{i \neq j} \pi_i P_{ij}, \quad j \geq 0$$

- $\pi_j P_{ji}$ is the frequency of transitions from j to i

$$\begin{pmatrix} \text{Frequency of} \\ \text{transitions out of } j \end{pmatrix} = \begin{pmatrix} \text{Frequency of} \\ \text{transitions into } j \end{pmatrix}$$

- Intuition: 1) j visited infinitely often; 2) for each transition out of j there must be a subsequent transition into j with probability 1

Global Balance Equations (contd.)

- Alternative Form of GBE

$$\sum_{j \in S} \pi_j \sum_{i \notin S} P_{ji} = \sum_{i \notin S} \pi_i \sum_{j \in S} P_{ij}, \quad S \subseteq \{0, 1, 2, \ldots\}$$

- *If a probability distribution satisfies the GBE, then it is the unique stationary distribution of the Markov chain*
- Finding the stationary distribution:
  - Guess distribution from properties of the system
  - Verify that it satisfies the GBE
  - ☺ Special structure of the Markov chain simplifies task

Global Balance Equations – Proof

First form:

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \text{ and } \sum_{i=0}^{\infty} P_{ji} = 1 \quad \Rightarrow$$

$$\pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \Leftrightarrow \pi_j \sum_{i \neq j} P_{ji} = \sum_{i \neq j} \pi_i P_{ij}$$

Second form:

$$\pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \Rightarrow \sum_{j \in S} \pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{j \in S} \sum_{i=0}^{\infty} \pi_i P_{ij} \Rightarrow$$

$$\sum_{j \in S} \pi_j \left(\sum_{i \in S} P_{ji} + \sum_{i \notin S} P_{ji} \right) = \sum_{j \in S} \left(\sum_{i \in S} \pi_i P_{ij} + \sum_{i \notin S} \pi_i P_{ij} \right) \Rightarrow$$

$$\sum_{j \in S} \pi_j \sum_{i \notin S} P_{ji} = \sum_{i \notin S} \pi_i \sum_{j \in S} P_{ij}$$

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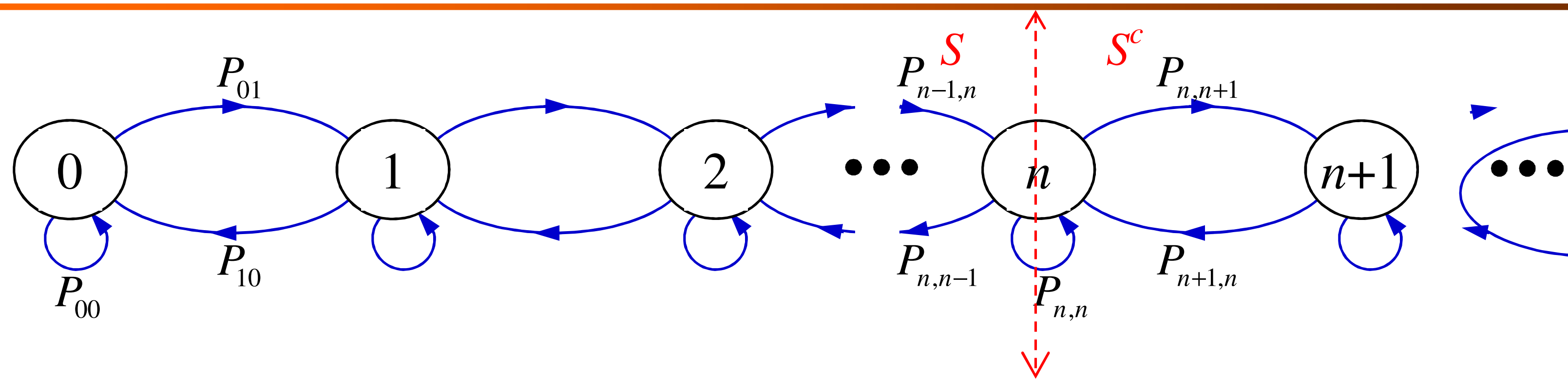
Birth-Death Process



- One-dimensional Markov chain with transitions only between neighboring states: $P_{ij}=0$, if $|i\text{-}j|>1$
- Detailed Balance Equations (DBE)

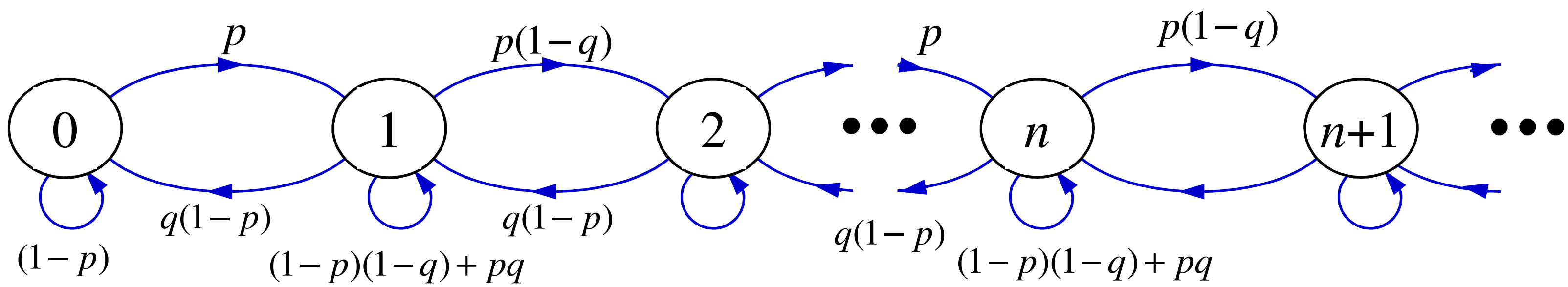
$$\pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n} \qquad n = 0,1,\ldots$$

- Proof: GBE with $S=\{0,1,\ldots,n\}$ give:

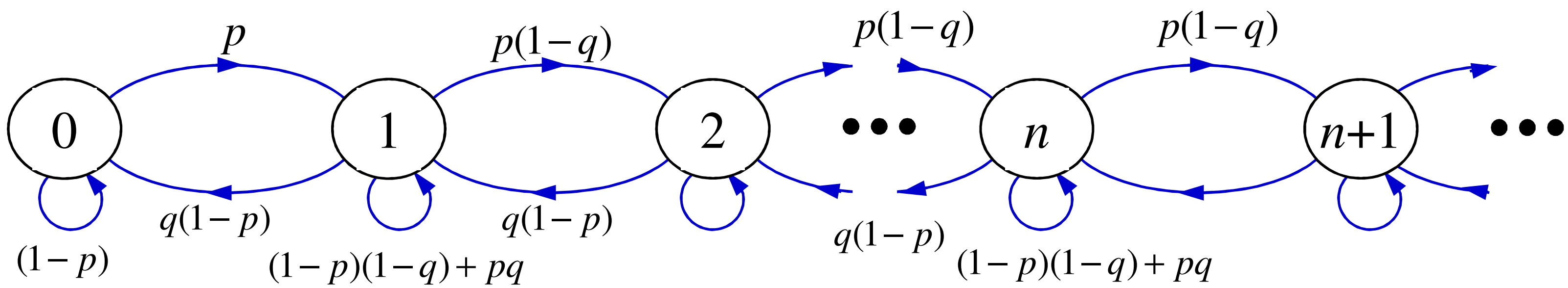
$$\sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_j P_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_i P_{ij} \Rightarrow \pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n}$$

Example: Discrete-Time Queue

- In a time-slot, one packet arrival with probability p or zero arrivals with probability $1\text{-}p$
- In a time-slot, the packet in service departs with probability q or stays with probability $1\text{-}q$
- Independent arrivals and service times
- State: number of packets in system



Example: Discrete-Time Queue (contd.)



$$\pi_0 p = \pi_1 q(1-p) \Rightarrow \pi_1 = \frac{p/q}{1-p}\pi_0$$

$$\pi_n p(1-q) = \pi_{n+1} q(1-p) \Rightarrow \pi_{n+1} = \frac{p(1-q)}{q(1-p)}\pi_n, \ \ n \geq 1$$

Define: $\rho \equiv p/q, \ \ \alpha \equiv \dfrac{p(1-q)}{q(1-p)}$

$$\begin{cases} \pi_1 = \dfrac{\rho}{1-p}\pi_0 \\ \pi_{n+1} = \alpha\pi_n, \ \ n \geq 1 \end{cases} \Rightarrow \ \pi_n = \alpha^{n-1}\frac{\rho}{1-p}\pi_0, \ \ n \geq 1$$

Example: Discrete-Time Queue (contd.)

- Having determined the distribution as a function of π_0

$$\pi_n = \alpha^{n-1} \frac{\rho}{1-p} \pi_0, \quad n \geq 1$$

How to calculate the normalization constant π_0?

- Probability conservation law:

$$\sum_{n=0}^{\infty} \pi_n = 1 \Rightarrow \pi_0 = \left[1 + \frac{\rho}{1-p} \sum_{n=1}^{\infty} \alpha^{n-1} \right]^{-1} = \left[1 + \frac{\rho}{(1-p)} \frac{1}{(1-\alpha)} \right]^{-1}$$

- Noting that

$$(1-p)(1-\alpha) = (1-p) \frac{q(1-p) - p(1-q)}{q(1-p)} = \frac{q-p}{q} = 1-\rho$$

$$\begin{cases} \pi_0 = 1-\rho \\ \pi_n = \rho(1-\alpha)\alpha^{n-1}, \quad n \geq 1 \end{cases}$$

Detailed Balance Equations

- General case:

$$\pi_j P_{ji} = \pi_i P_{ij} \quad i, j = 0,1,\ldots$$

- Need NOT hold for every Markov chain
- If hold, it implies the GBE; greatly simplify the calculation of stationary distribution

<u>Methodology:</u>

- Assume DBE hold – have to guess their form
- Solve the system defined by DBE and $\Sigma_i \pi_i = 1$
  - If system is inconsistent, then DBE does not hold
  - If system has a solution $\{\pi_i: i=0,1,\ldots\}$, then it is the unique stationary distribution

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Generalized Markov Chains

- Markov chain on a set of states {0,1,…}, that whenever enters state i
  - The next state that will be entered is j with probability P_{ij}
  - Given that the next state entered will be j, the time it spends at state i until the transition occurs is a RV with distribution F_{ij}
- $\{Z(t): t \geq 0\}$ describing the state of the chain at time t: *Generalized Markov chain*, or *Semi-Markov* process
  - Does GMC have the Markov property?
    - Future depends on 1) the present state, and 2) the length of time the process has spent in this state

Generalized Markov Chains (contd.)

- T_i: time process spends at state i, before making a transition – *holding time*
- Probability distribution function of T_i

$$H_i(t) = P\{T_i \le t\} = \sum_{j=0}^{\infty} P\{T_i \le t \mid \text{next state } j\} P_{ij} = \sum_{j=0}^{\infty} F_{ij}(t) P_{ij}$$

$$E[T_i] = \int_0^{\infty} t \, dH_i(t)$$

- T_{ii}: time between successive transitions to i
- X_n is the n^{th} state visited. $\{X_n: n=0,1,\ldots\}$
  - Is a Markov chain: embedded Markov chain
  - Has transition probabilities P_{ij}
- Semi-Markov process *irreducible*: if its embedded Markov chain is irreducible

Limit Theorems

Theorem 3: given an irreducible semi-Markov process w/ $\mathrm{E}[T_{ii}] < \infty$

- For any state j, the following limit

$$p_j = \lim_{t\to\infty} P\{Z(t) = j \mid Z(0) = i\}, \quad i = 0,1,2,\ldots$$

  exists and is independent of the initial state.

$$p_j = \frac{E[T_j]}{E[T_{jj}]}$$

- $T_j(t)$: time spent at state j up to time t

$$P\left\{ p_j = \lim_{t\to\infty} \frac{T_j(t)}{t} \mid Z(0) = i \right\} = 1$$

  - p_j is equal to *the proportion of time spent at state j*

Occupancy Distribution

<u>Theorem 4:</u> given an irreducible semi-Markov process where $E[T_{ii}] < \infty$, and the embedded Markov chain is ergodic w/ stationary distribution π

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij},\ j \geq 0; \quad \sum_{i=0}^{\infty} \pi_i = 1$$

- then, with probability 1, the occupancy distribution of the semi-Markov process

$$p_j = \frac{\pi_j E[T_j]}{\sum_i \pi_i E[T_i]}, \quad j = 0, 1, \ldots$$

  - π_j: proportion of transitions into state j
  - $E[T_j]$: mean time spent at j
  - Probability of being at j is proportional to $\pi_j E[T_j]$

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Continuous-Time Markov Chains (def.?)

Continuous-time process $\{X(t)\colon t \geq 0\}$ taking values in {0,1,2,…}.

Whenever it enters state i

- Time it spends at state i is *exponentially distributed* with parameter ν_i
- When it leaves state i, it enters state j with probability P_{ij}, where $\Sigma_{j \neq i} P_{ij} = 1$

- Continuous-time Markov chain is a semi-Markov process with

$$F_{ij}(t) = 1 - e^{-\nu_i t}, \quad i, j = 0, 1, \ldots$$

- Exponential holding time => a continuous-time Markov chain has the Markov property

Continuous-Time Markov Chains

- When at state i, the process makes transitions to state $j \neq i$ with rate:

$$q_{ij} \equiv \nu_i P_{ij}$$

- Total rate of transitions out of state i

$$\sum_{j \neq i} q_{ij} = \nu_i \sum_{j \neq i} P_{ij} = \nu_i$$

- Average time spent at state i before making a transition:

$$E[T_i] = 1/\nu_i$$

Occupancy Probability

- A continuous-time Markov chain *is irreducible and regular*, if
  - Embedded Markov chain is irreducible
  - Number of transitions in a finite time interval is finite with probability 1
- From Theorem 3: for any state j, the limit

$$p_j = \lim_{t\to\infty} P\{X(t) = j \mid X(0) = i\}, \quad i = 0, 1, 2, \ldots$$

  exists and is independent of the initial state
  - p_j is the steady-state *occupancy probability* of state j
  - p_j is equal to the proportion of time spent at state j

Global Balance Equations

- Two possibilities for the occupancy probabilities:
  - $p_j = 0$, for all j
  - $p_j > 0$, for all j, and $\Sigma_j p_j = 1$
- Global Balance Equations

$$p_j \sum_{i \neq j} q_{ji} = \sum_{i \neq j} p_i q_{ij}, \quad j = 0, 1, \ldots$$

  - Rate of transitions out of j = rate of transitions into j
  - If a distribution $\{p_j : j = 0, 1, \ldots\}$ satisfies GBE, then it is the *unique* occupancy distribution of the Markov chain
- Alternative form of GBE:

$$\sum_{j \in S} p_j \sum_{i \notin S} q_{ji} = \sum_{i \notin S} p_i \sum_{j \in S} q_{ij}, \quad S \subseteq \{0, 1, \ldots\}$$

Detailed Balance Equations

- Detailed Balance Equations

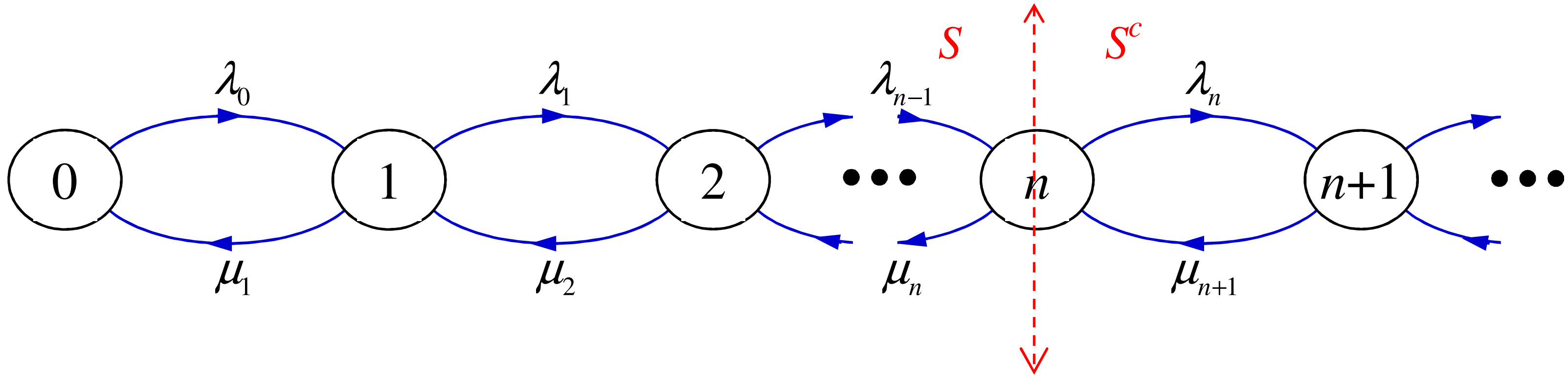
$$p_j q_{ji} = p_i q_{ij}, \quad i, j = 0, 1, \ldots$$

☺ Simplify the calculation of the stationary distribution

☹ Need not hold for every Markov chain

- Examples: birth-death processes, and reversible Markov chains

Birth-Death Process



- Transitions only between neighboring states

$$q_{i,i+1} = \lambda_i, \quad q_{i,i-1} = \mu_i, \quad q_{ij} = 0, \ |i-j| > 1$$

- Detailed Balance Equations

$$\lambda_n p_n = \mu_{n+1} p_{n+1}, \quad n = 0,1,\ldots$$

- Proof: GBE with $S = \{0,1,\ldots,n\}$ give:

$$\sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_j q_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_i q_{ij} \Rightarrow \lambda_n p_n = \mu_{n+1} p_{n+1}$$

Birth-Death Process

$$\mu_n p_n = \lambda_{n-1} p_{n-1} \Rightarrow$$

$$p_n = \frac{\lambda_{n-1}}{\mu_n} p_{n-1} = \frac{\lambda_{n-1}}{\mu_n}\frac{\lambda_{n-2}}{\mu_{n-1}} p_{n-2} = \ldots = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} p_0 = p_0 \prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}$$

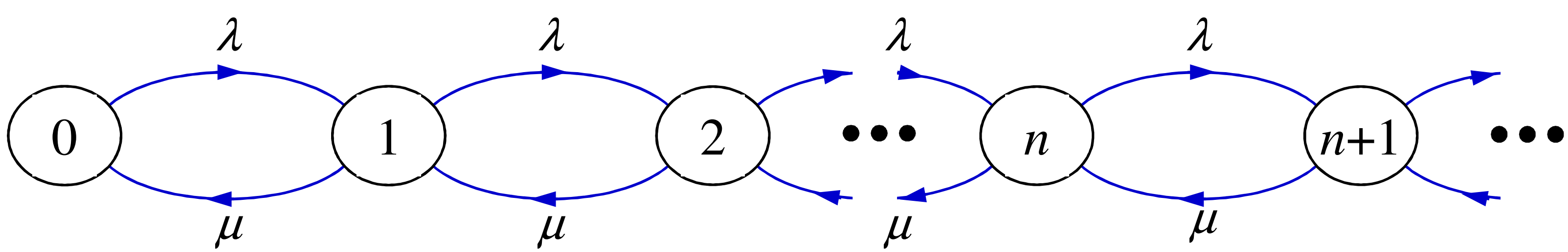
$$\sum_{n=0}^{\infty} p_n = 1 \Leftrightarrow p_0\left[1+\sum_{n=1}^{\infty}\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right] = 1 \Leftrightarrow p_0 = \left[1+\sum_{n=1}^{\infty}\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right]^{-1}, \text{ if } \sum_{n=1}^{\infty}\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}} < \infty$$

- Use DBE to determine state probabilities as a function of p_0
- Use the probability conservation law to find p_0

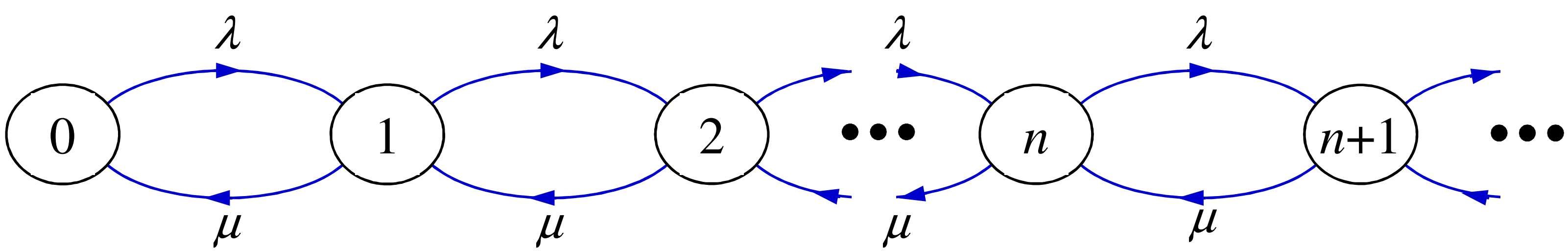
- Using DBE in solving problems:
  - Prove that DBE hold, or
  - Justify validity (e.g. reversible process), or
  - Assume they hold – have to guess their form – and solve system

M/M/1 Queue

- Arrival process: Poisson with rate λ
- Service times: iid, exponential with parameter μ
- Service times and interarrival times: independent
- Single server
- Infinite waiting room
- N(t): Number of customers in system at time t (state)



M/M/1 Queue



- Birth-death process → DBE

$$\mu p_n = \lambda p_{n-1} \Rightarrow$$

$$p_n = \frac{\lambda}{\mu} p_{n-1} = \rho p_{n-1} = \ldots = \rho^n p_0$$

- Normalization constant

$$\sum_{n=0}^{\infty} p_n = 1 \Leftrightarrow p_0 \left[1 + \sum_{n=1}^{\infty} \rho^n \right] = 1 \Leftrightarrow p_0 = 1 - \rho \;, \;\; \text{if } \rho < 1$$

- Stationary distribution

$$p_n = \rho^n (1-\rho), \quad n = 0, 1, \ldots$$

The M/M/1 Queue

- Average number of customers

$$N = \sum_{n=0}^{\infty} n p_n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = (1-\rho)\rho\sum_{n=0}^{\infty} n\rho^{n-1}$$

$$\Rightarrow N = \rho(1-\rho)\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

- Applying Little's Theorem, we have

$$T = \frac{N}{\lambda} = \frac{1}{\lambda}\frac{\lambda}{\mu-\lambda} = \frac{1}{\mu-\lambda}$$

- Similarly, the average waiting time and number of customers in the queue is given by

$$W = T - \frac{1}{\mu} = \frac{\rho}{\mu-\lambda} \text{ and } N_Q = \lambda W = \frac{\rho^2}{1-\rho}$$

Summary

- Markov Chain

- Discrete-Time Markov Chains
- Calculating Stationary Distribution
- Global Balance Equations
- Birth-Death Process
  - Detailed Balance Equations

- Generalized Markov Chains
- Continuous-Time Markov Chains

Homework #8

- Problem 3.14 of R1

- Hints:
  - For a service system, the expected number of customers is finite if the service rate is greater than the customer arrival rate.
  - To solve the problem, think of how to model the system as a Markov process. You may also find Little's Theorem be of some use in solving the problem.

- Grading:
  - Overall points 100
    - 30 points for 3.14(a)
    - 70 points for 3.14(b)