Approximate Bandwidth Allocation for Fixed-Priority-Scheduled Periodic Resources (WSU-CS Technical Report Version)*

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Abstract

Recent research in compositional real-time systems has focused on determination of a component's real-time interface parameters. An important objective in interface-parameter determination is minimizing the bandwidth allocated to each component of the system while simultaneously guaranteeing component schedulability. With this goal in mind, in this paper we develop a fully-polynomial-time approximation scheme (FPTAS) for allocating bandwidth for sporadic task systems scheduled by fixed priority (e.g., deadline monotonic, rate monotonic) upon an Explicit-Deadline Periodic (EDP) resource. Our parametric algorithm takes the task system and an accuracy parameter $\epsilon > 0$ *as input, and returns a bandwidth which is guaranteed to be at most a factor* $(1 + \epsilon)$ *times the optimal minimum bandwidth required to successfully schedule the task system. By simulations over synthetically generated task systems, we observe a significant decrease in runtime and a small relative error when comparing our proposed algorithm with the exact algorithm and the sufficient algorithm.*

1 Introduction

Recent research in real-time systems has focused on designing frameworks for enabling component-based design in real-time systems. State-of-the-art frameworks for compositional real-time systems include [1, 9, 12, 25]. Component-based design is highly desirable due to its well-known benefits of reducing overall system complexity and enhancing system designers' understanding of the system. One of the major benefits of these systems is achieved by the goal of *component abstraction*, which hides the internal complexity and details of one component from developers of other components and only exposes information necessary to use the component via an interface. In most compositional frameworks, a component uses a *real-time interface* to communicate with the other components of the system. A component specifies its resource requirements to meet its real-time constraints by the attribute *interface bandwidth*. Thus, an important design issue of these compositional frameworks is addressing the problem *minimization of interface bandwidth* (MIB-RT).

One simple, yet flexible, real-time compositional framework is the *explicit-deadline periodic resource* (EDP) model [25]. An EDP resource Ω is characterized by a three-tuple (Π, Θ, Δ) where Π is referred to as the *period of repetition*, Θ is the *capacity*, and Δ is the *relative deadline*. The interpretation of such a resource is that a component C executed upon Ω is guaranteed Θ units of processing resource supply for successive Π-length intervals (given some initial starting time). Furthermore, the Θ units of resource supply must be provided within Δ $(\leq \Pi)$ time units after the start of the Π-length interval. The interface bandwidth of C for this framework is $\frac{\Theta}{\Pi}$. A system-level scheduling algorithm allocates the processor time among the different periodic resources that share the same processor, such that each resource receives (for every period) aggregate processor time equivalent to its capacity. A component's tasks are then hierarchically scheduled by a component-level scheduling algorithm upon the processing time supplied to resource Ω.

In this paper, we obtain solutions to MIB-RT for an EDP resource when components use fixed-priority as the component-level scheduling algorithm. (The system-level scheduling algorithm is not considered for this paper). Specifically, we consider the problem of determining the optimal choice of capacity parameter (i.e., Θ) for an EDP resource Ω with a fixed period Π and deadline Δ for component C. Algorithms exist for determining Π and Δ by searching over possible values and using a

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capacity-determination algorithm as a subroutine (e.g., see [10,13]); thus, since the search space may be quite large, efficient capacity-determiniation algorithms are necessary.

The MIB-RT problem for fixed-priority periodic resource model has been previously studied. An exact solution based on exact schedulability techniques for uniprocessor real-time systems ([6,16]) has been proposed by Easwaran et al. [11]. There is also a $O(n)$-time sufficient solution to MIB-RT for periodic resource (Π equals Δ) by Shin and Lee [25]. The exact resource allocation is computationally expensive (pseudo-polynomial in this case) and thus might be impractical for algorithms that search for optimal values of Π and Δ. On the other hand, though the sufficient resource allocation has lower (linear) computational complexity, these algorithms might provide over-estimated resource allocations and induce lower system utilization. This might be impractical for developing real-time systems in which resources are very scarce. However, in many real-time systems where tasks may be added or removed dynamically, it is important to provision resources efficiently at run-time and an efficient allocation algorithm is desirable. Our goal is to design an algorithm which is computationally efficient on real-time guarantee verification as well as to provide the system designer control over accuracy of resource allocation.

In our prior work [15], we devised approximate bandwidth allocation algorithm for EDP resource with component level scheduling algorithm EDF. In this paper we extend those results for fixed priority scheduled components. However, the compositional results for EDF does not directly apply for fixed priority, as we have to do maximum response time analysis for each task; this fundamentally differs from the demand-based approach of [15].

§**Our Contribution.** For EDP resource model with sporadic tasks [20] as components, we develop a parametric approximation algorithm that addresses the current gap between computationally-expensive, exact solutions and computationally-inexpensive, sufficient solutions for MIB-RT problem. We claim the following.

> Given Π, Δ, task system τ, and accuracy parameter $\epsilon > 0$, let $\Theta^*(\Pi, \Delta, \tau)$ be the optimal minimum capacity for τ to be fixed-priority-schedulable upon EDP resource $\Omega^* = (\Pi, \Theta^*(\Pi, \Delta, \tau), \Delta)$. Our algorithm returns $\widehat{\Theta}$ for the given parameters where $\Theta^*(\Pi, \Delta, \tau) \leq \widehat{\Theta} \leq (1+\epsilon) \cdot \Theta^*(\Pi, \Delta, \tau)$. Furthermore, the time complexity of our algorithm is polynomial in the number of tasks in τ and $\frac{1}{\epsilon}$.

In other words, our algorithm is a fully-polynomial-time approximation scheme (FPTAS) for the MIB-RT problem with the *approximation ratio* $(1 + \epsilon)$. This implies that the system designer can pre-specify an arbitrary level of accuracy in obtaining solution to MIB-RT with the tunable algorithm. We also validate our algorithm by means of simulation over randomly generated task systems.

§**Organization.** The remainder of the paper is organized as follows. In Section 2, we briefly review the current literature on compositional real-time frameworks and MIB-RT problem for fixed-priority. In Section 3, we provide necessary notations required for the rest of the paper. In Section 4, we present an approximate algorithm for bandwidth allocation, and prove its correctness. Then, we give the approximation ratio results for the proposed algorithm in Section 5. Simulation results comparing our algorithm with both previously-known exact and sufficient algorithms are given in Section 6. Finally, we conclude with discussion and future direction of this research in Section 7.

2 Related Work

In this section, we give a very high-level overview of some of the prior work on MIB-RT for compositional real-time systems. The concept of compositional real-time system was first introduced by Deng and Liu [9] in their work *real-time open environments* and Rajkumar et al. [21] in their work *resource kernels*. Since then, researchers have proposed many different real-time compositional models and studied the MIB-RT problem of the proposed models. Two of the well studied approaches in the literature are the partition-based models and the demand-based models. Feng and Mok [12] proposed the concept of *temporal partitions* to support hierarchical sharing of a processing resource. Shin et al. [23,25] proposed the related *periodic resource model* to characterize the supply guaranteed to any component in compositional system. For the temporal partition models where components are scheduled by fixed-priority, Lipari and Bini [18] proposed exact, pseudo-polynomial time algorithm for MIB-RT, and Almeida and Pedreiras [3] proposed sufficient, polynomial-time bandwidth allocation techniques. In the demand-based models, components of the system are characterized by processor-demand curves, which describe the minimum amount of processing required by a component over any time interval. Wandeler and Thiele [26] proposed the concept of *interface-based design* in which real-time calculus [8] is used to compute demand curves and service curves for each component in a compositional real-time system. In another demand-based model known as *hierarchical event stream model*, Albers et al. [1] have developed parametric algorithms for MIB-RT (without known approximation ratios). Thus, for a variety of both partition-based models, relatively efficient, sufficient algorithms for MIB-RT have been proposed; however, the existence of any work on obtaining polynomial-time algorithms with constant-factor approximation ratios where components are scheduled by fixed priority is unknown. In our preliminary work [15], we obtained such ratios for the periodic resource model where components are scheduled by dynamic-priority (EDF). In this paper, our aim is to extend those results by developing an FPTAS for EDP framework where components are scheduled by fixed-priority (DM or RM). The algorithm of [13] may be used in conjunction with the results of this paper to find an optimal period.

3 Models and Notation

In this section, we present background and notation for the task model, workload functions, and periodic resource model that we use throughout the paper.

§**Sporadic Task Model.** A **sporadic task** $\tau_i = (e_i, d_i, p_i)$ is characterized by a *worst-case execution requirement* e_i, a *(relative) deadline* d_i, and a *minimum inter-arrival separation* p_i. Such a sporadic task generates a potentially infinite sequence of jobs, with successive job-arrivals separated by at least p_i time units. Each job has a worst-case execution requirement equal to e_i and a deadline that occurs d_i time units after its arrival time. A *sporadic task system* $\tau = \{\tau_1, \ldots, \tau_n\}$ is a collection of n such sporadic tasks. We will assume that each task $\tau_i \in \tau$ has $d_i \leq p_i$; such sporadic task systems are known as *constrained-deadline* sporadic task systems. The *task utilization* for a sporadic task τ_i is defined as $u_i = e_i/p_i$. The system utilization is denoted by $U(\tau) = \sum_{\tau_i \in \tau} u_i$. We will assume that $U(\tau) \leq 1$; otherwise, task system τ cannot be scheduled (by any algorithm) to meet all deadlines upon a dedicated preemptive uniprocessor.

We will assume that each task has a fixed priority and tasks are indexed in non-increasing priority order. That is, τ_i has higher (or equal) priority than τ_j, if and only if, $i \leq j$. As tasks generate jobs, each job inherits the priority of its generating task (i.e., all jobs generated by task τ_i have the same priority as τ_i). For this paper, we assume each component uses *fixed-priority scheduling* as the component-level scheduling algorithm. Whenever component C is allocated the processor, C executes the highest-priority job with remaining execution; ties are broken in favor of the job generated from the lower task index. One noteworthy fixed-priority scheduling algorithm is *deadline monotonic* (DM) [17], which assigns each task a priority equal to the inverse of its relative deadline (i.e., tasks with shorter relative deadlines have priority greater than tasks with longer relative deadlines). DM is known to an optimal fixed-priority uniprocessor scheduling algorithm for constrained-deadline sporadic tasks in the following sense; if a constrained-deadline sporadic task system is schedulable upon a single processor by a fixed-priority assignment, then it will also meet all deadlines under the DM priority assignment.

§**Workload Functions.** To determine the schedulability of a sporadic task system, it is often useful to quantify the maximum amount of execution time requested by the task in its worst-case phasing over any given interval. For sporadic task systems, it is known that the worst-case phasing is the *synchronous arrival sequence*. The synchronous arrival sequence occurs when all tasks of a sporadic task system release jobs at the same time instant and subsequent jobs as soon as permissible. Researchers [16] have derived the *request-bound function*, as defined below.

Definition 1 (Request-Bound Function) *For any $t > 0$ and sporadic task τ_i, the* **request-bound function** *(*RBF*) quantifies the maximum cumulative execution requests that could be generated by jobs of τ_i arriving within a contiguous time-interval of length t. It has been shown that for sporadic tasks,* RBF *can be calculated as follows [16].*

$$\mathsf{RBF}(\tau_i, t) \stackrel{\text{def}}{=} \left\lceil \frac{t}{p_i} \right\rceil \cdot e_i. \tag{1}$$

Figure 1 shows the request-bound function for a sporadic task τ_i, which is a right continuous function with discontinuities at time points of the form $t \equiv a \cdot p_i$ where $a \in \mathbb{N}$. The *cumulative request-bound function* for task τ_i is defined as follows:

$$W_i(t) \stackrel{\text{def}}{=} e_i + \sum_{j=1}^{i-1} \mathsf{RBF}(\tau_j, t). \tag{2}$$

Audsley *et al.* [5] have given a necessary and sufficient condition for sporadic task system τ to be fixed-priority-schedulable upon a preemptive uniprocessor platform of unit speed: $\exists t \in (0, d_i]$ such that $W_i(t) \leq t, \forall i$. Furthermore, it has also been shown [4] that this condition needs to be verified at only time points in the following *ordered* set:

$$S_i(\tau) \stackrel{\text{def}}{=} \left\{ t = b \cdot p_a : a = 1, \ldots, i; b = 1, \ldots, \left\lfloor \frac{d_i}{p_a} \right\rfloor \right\} \cup \{d_i\}. \tag{3}$$

The above set is known as the *testing set* for sporadic task τ_i. The size of this set may be as large as $\sum_{j=1}^{i} \left\lfloor \frac{d_i}{p_j} \right\rfloor$ which is dependent on the task periods, and thus requires pseudo-polynomial time feasibility test. Fisher and Baruah [14] proposed

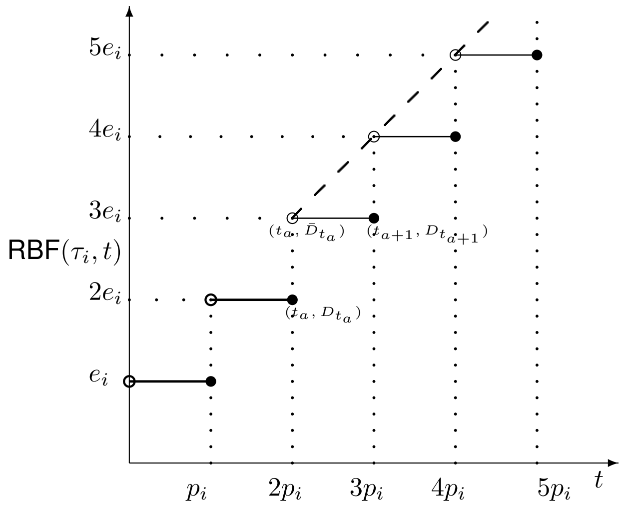

Figure 1. The step function denotes a plot of $\mathsf{RBF}(\tau_i, t)$ as a function of t. The dashed line represents the function $\delta(\tau_i, t)$, approximating $\mathsf{RBF}(\tau_i, t)$. $\delta(\tau_i, t)$ is equal to $\mathsf{RBF}(\tau_i, t)$ for all $t \leq (k-1)p_i$. The value of k equals three in the above graph.

the following approximation to RBF (inspired by a similar approximation for EDF due to Albers and Slomka [2]) to reduce the number of points in the testing set.

$$\delta(\tau_i, t, k) \stackrel{\text{def}}{=} \begin{cases} \mathsf{RBF}(\tau_i, t), & \text{if } t \leq (k-1)p_i \\ e_i + \frac{t \cdot e_i}{p_i}, & \text{otherwise.} \end{cases} \tag{4}$$

This function tracks RBF for exactly $k-1$ steps and after the $k-1$-th step, it uses linear interpolation of subsequent discontinuous points of RBF (with slope equal to u_i). The steps in Figure 1 correspond to $\mathsf{RBF}(\tau_i, t, k)$, and the thick steps and the sloped-dashed line correspond to $\delta(\tau_i, t, k)$. The approximate cumulative request bound function is defined as follows:

$$\widehat{W}_i(t) \stackrel{\text{def}}{=} e_i + \sum_{j=1}^{i-1} \delta(\tau_j, t, k). \tag{5}$$

For any fixed $k \in \mathbb{N}^+$, Fisher and Baruah [14] showed that if for all $\tau_i \in \tau$ there exists a $t \in (0, d_i]$ such that $\widehat{W}_i(t) \leq t$ then the sporadic task system τ is static priority schedulable upon a preemptive uniprocessor platform of unit speed. The testing set for this condition is as follows:

$$\widehat{S}_i(\tau, k) \stackrel{\text{def}}{=} \{t = b \cdot p_a \mid a = 1, \ldots, i-1; b = 1, \ldots, k-1; t \in (0, d_i]\} \cup \{d_i\} \cup \{0\} \tag{6}$$

Let t_a, t_{a+1} denote any pair of consecutive values in the above ordered set.

Next, we give the relation between the request bound function RBF and the approximate request bound function δ.

Lemma 1 (from [14]) *Given a fixed integer $k \in \mathbb{N}^+$, $\mathsf{RBF}(\tau_i, t) \leq \delta(\tau_i, t, k) \leq \left(\frac{k+1}{k}\right) \mathsf{RBF}(\tau_i, t)$ for all $\tau_i \in \tau$ and $t \in \mathbb{R}_{\geq 0}$.*

We will use this lemma in our approximation algorithm (Section 5).

Next, we define notation to represent the discontinuous line segments of the cumulative request bound function ($\widehat{W}_i$). Let $\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$ be a line segment in the Euclidean space, $\mathbb{R}^2$, originating at point $(t_a, \bar{D}_{t_a}) \in \mathbb{R}^2$ and ending at point $(t_{a+1}, D_{t_{a+1}}) \in \mathbb{R}^2$ with slope $\alpha \geq 0$; more formally,

$$\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 \mid (x \in [t_a, t_{a+1}]) \wedge (y = \alpha(x - t_a) + \bar{D}_{t_a}))\}. \tag{7}$$

Please note the term α is included in the notation for convenience only; it is possible to determine the slope from points $(t_a, \bar{D}_{t_a})$ and $(t_{a+1}, D_{t_{a+1}})$ alone. We denote any point in the line segment by $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$.

The connection between $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ and $\widehat{W}_i$ is as follows. Consider a time $t_a \in \widehat{S}_i(\tau, k)$. Define D_{t_a} to be request bound function at time t_a, that is $\widehat{W}_i(t_a)$ (Figure 1). At time t_a, some set of tasks with priority greater τ_i have job arrivals in the synchronous arrival sequence. Let $r_i(t)$ be the sum of the executions of these tasks. Formally,

$$r_i(t) \stackrel{\text{def}}{=} \sum_{\tau_j \in \tau : (j<i) \wedge (p_j \; divides \; t)} e_j. \tag{8}$$

At time t_a there is a discontinuity in the function $\widehat{W}_i$ in which $\widehat{W}_i$ increases by $r_i(t_a)$ and then is linear until the next discontinuity in $\widehat{W}_i$ (i.e., at time $t_{a+1} \in \widehat{S}_i(\tau, k)$). Thus, $\widehat{W}_i$ is a line segment from t_a to t_{a+1} with slope equal to the total utilization of all task τ_j such that $j < i$ and $t_a \geq (k-1)p_j$. We denote $\bar{D}_{t_a}$ by the sum of request bound function at time point t_a and job release at time t_a, that is, $\bar{D}_{t_a} = D_{t_a} + r_i(t_a)$. Finally, observe that

$$\alpha \stackrel{\text{def}}{=} \sum_{\tau_j \in \tau : (t_a \geq (k-1)p_j) \wedge (j<i)} u_j. \tag{9}$$

From the above definitions of t_a, t_{a+1}, D_{t_a}, $\bar{D}_{t_a}$, $D_{t_{a+1}}$, and α, it is straightforward to verify that the line segment $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ is equivalent to $(t, \widehat{W}_i(t))$ for all $t \in (t_a, t_{a+1}]$. The exception is at t_a where $\bar{D}_{t_a}$ is not equal to $\widehat{W}_i(t_a)$; this difference exists for the notational and algebraic convenience throughout Section 4. From the definitions of $t_a, t_{a+1}, D_{t_a}, \bar{D}_{t_a}, D_{t_{a+1}}$, and α, the following lemma is apparent.

Lemma 2 *For any consecutive pairs of values $(t_a, t_{a+1}) \in \widehat{S}_i(\tau, k)$, $\widehat{W}_i(t) \leq D_t$ for all $(t, D_t) \in \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$.*

§**Explicit-Deadline Periodic (EDP) Resource Model.** An EDP resource, denoted by $\Omega = (\Pi, \Theta, \Delta)$, guarantees that a component C executed upon resource Ω will receive at least Θ units of execution between successive time points in $\{t \equiv t_0 + \ell\Pi \mid \ell \in \mathbb{N}\}$ where t_0 is some initial service start-time t_0 for the periodic resource. Furthermore, the Θ units of service must occur Δ units after each successive time point in the aforementioned set. Obviously, $\Theta \leq \Delta$; for this paper, we will make the simplifying assumption that $\Delta \leq \Pi$, as well. Furthermore, we will assume in this paper that each component C is a sporadic task system τ scheduled by fixed priority upon Ω. (From now on, we use τ in the context of component C).

Definition 2 (Supply-Bound Function) *For any $t > 0$, the* **supply-bound function** *(sbf) quantifies the minimum execution supply that a component executed upon periodic resource Ω may receive over any interval of length t. It is defined as follows [11]:*

$$\mathsf{sbf}(\Omega, t) \stackrel{\text{def}}{=} \begin{cases} y_\Omega \Theta + \max\left(0, t - x_\Omega - y_\Omega \Pi\right), & \text{if } t \geq \Delta - \Theta \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

where $y_\Omega \stackrel{\text{def}}{=} \left\lfloor \frac{t-(\Delta-\Theta)}{\Pi} \right\rfloor$ *and* $x_\Omega \stackrel{\text{def}}{=} (\Pi + \Delta - 2\Theta)$.

We denote the optimal minimum capacity for Ω given task system τ by $\Theta^*(\Pi, \Delta, \tau)$. We use the concept of *ℓ-feasibility region* of Ω similar to [15] to define the region under the ℓ-th step of sbf. For our convenience, we redefine ℓ-feasibility region as follows:

Definition 3 (ℓ-Feasibility Region of Ω)

$$\mathcal{F}_\ell(\Pi, \Delta, \Theta) \stackrel{\text{def}}{=} \left\{ (t, D_t) \in \mathbb{R}^2_{\geq 0} \;\middle|\; \begin{array}{cc} & \left(\Theta \geq \frac{D_t - t + \ell\Pi + \Delta}{\ell + 1}\right) \\ \wedge & \left(\Theta \geq \frac{D_t}{\ell}\right) \end{array} \right\}. \tag{11}$$

Figure 2 shows graphical depiction of the supply bound function sbf for EDP resource Ω. The shaded region in the figure corresponds to the ℓ-feasibility region for some step $\ell \in \mathbb{N}$ of the sbf.

The following theorem states the exact schedulability condition for EDP resource Ω, where task system is scheduled using fixed priority scheduling algorithm [11, 23, 24]. It says that for the task system τ to be schedulable with EDP resource, each task τ_i in τ must have a fixed point t before its deadline at which the cumulative request bound function for τ_i is less than the supply provided to the system at that point.

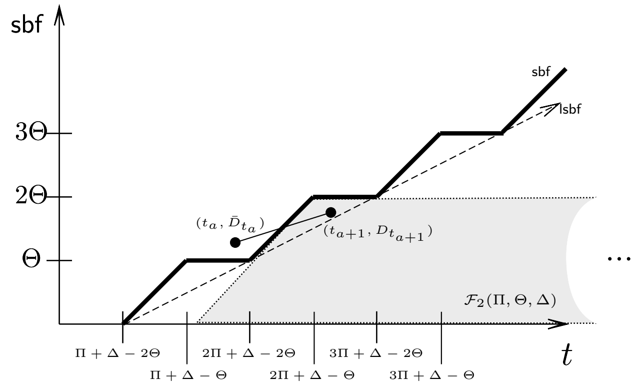

Figure 2. The solid line "step" function is sbf for Ω. The shaded region represents the ℓ-feasibility region for $\ell = 2$, containing a part of the line segment $\left\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\right\rangle$.

Theorem 1 (from [11]) *A sporadic task system τ is fixed priority schedulable upon an EDP resource $\Omega = (\Pi, \Theta, \Delta)$, if and only if,*

$$(\forall i,\ \exists t \in (0, d_i] : W_i(t) \leq \mathsf{sbf}(\Omega, t)) \bigwedge \left(U(\tau) \leq \frac{\Theta}{\Pi} \right) \tag{12}$$

In the next section we present an approximate algorithm to obtain minimum capacity for EDP resource when the component-level scheduling algorithm is fixed priority for the task system τ. We consider fixed period (Π) and deadline (Δ) for the EDP resource Ω.

4 An Algorithm for Determining Minimum Capacity

In Figure 3, we present the pseudocode for our algorithm, FPMINIMUMCAPACITY. Given task system τ and an EDP resource with Π and Δ as input, the algorithm returns approximate minimum capacity to correctly schedule the task system with the resource. The approximation parameter of the algorithm is the input $k \in \mathbb{N}^+ \, (k = \lceil \frac{1}{\epsilon} \rceil)$. For some fixed k input, the algorithm returns the approximate minimum capacity; if k is equal to ∞, it returns exact minimum capacity. If FPMINIMUMCAPACITY returns a value $\Theta^{\min}$ that does not exceed Δ, then τ can be fixed-priority scheduled to meet all deadlines upon $\Omega = (\Pi, \Delta, \Theta^{\min})$. Note that the approximate capacity $\Theta^{\min}$ can be at most $(1 + \epsilon)$ times the exact capacity. If FPMINIMUMCAPACITY returns a capacity greater than Δ, then our algorithm cannot guarantee τ can be scheduled on any Ω with parameters Π and Δ. (Unless $k = \infty$, the algorithm is an approximation, and, thus, a returned capacity greater than Δ does not necessarily imply infeasibility of τ).

In our proposed algorithm, the objective is to compute minimum capacity $\Theta^{\min}$ for a task system τ such that τ is fixed-priority schedulable under EDP resource model. For each task $\tau_i \in \tau$, we find minimum capacity $\Theta_i^{\min}$ such that there exists a fixed point $t \in (0, d_i]$ at which the supply bound function sbf exceeds the cumulative request bound function $\widehat{W}_i(t)$ (Theorem 1). To calculate $\Theta_i^{\min}$, we determine, for each consecutive pair of values (t_a, t_{a+1}) in the testing set $\widehat{S}_i(\tau, k)$, the minimum capacity $\Theta_{t_a}^{\min}$ required to guarantee that the line segment $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ is beneath $\mathsf{sbf}((\Pi, \Theta_{t_a}^{\min}, \Delta), t)$ for some $t \in (t_a, t_{a+1}]$. Since $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ is equivalent to $\widehat{W}_i$ for all $t \in (t_a, t_{a+1}]$, this implies that there exist a $t \in (t_a, t_{a+1}]$ such that $\widehat{W}_i(t) \leq \mathsf{sbf}((\Pi, \Theta_{t_a}^{\min}, \Delta), t)$. To determine $\Theta_{t_a}^{\min}$, we take specific steps of the sbf (denote a selected step by ℓ) and determine the minimum Θ_ℓ such that some point of the line segment is below the ℓ-feasibility region with capacity Θ_ℓ. Each Θ_ℓ for (t_a, t_{a+1}) is set in lines 9, 10, 11 and 12. The following functions are used to determine the values of Θ_ℓ in our algorithm.

$$\begin{aligned}
\Phi_1(\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle, \ell, \Pi, \Delta) &\stackrel{\text{def}}{=} \tfrac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell + 1}, \\
\Phi_2(\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle, \ell, \Pi, \Delta) &\stackrel{\text{def}}{=} \frac{\bar{D}_{t_a}}{\ell}, \\
\Phi_3(\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle, \ell, \Pi, \Delta) &\stackrel{\text{def}}{=} \tfrac{\bar{D}_{t_a} + \alpha(\ell\Pi + \Delta - t_a)}{\ell + \alpha}.
\end{aligned} \tag{13}$$

We will also show that we only need to consider the integer values of ℓ given by the following equations.

FPMinimumCapacity(Π, Δ, τ, k)

1 $\Theta^{\min} \leftarrow U(\tau) \cdot \Pi$
2 **for each** $\tau_i \in \tau$
3 $\quad \Theta_i^{\min} \leftarrow \infty$
4 $\quad$ **for each** $(t_a, t_{a+1}) \in \widehat{S}_i(\tau, k)$ $\triangleright$ (In order)
5 $\quad\quad \bar{D}_{t_a} \leftarrow \hat{W}_i(t_a) + r_i(t_a)$
6 $\quad\quad D_{t_{a+1}} \leftarrow \hat{W}_i(t_{a+1})$
7 $\quad\quad \alpha \leftarrow \sum_{\tau_i \in \tau : t \geq d_i + (k-1)p_i} u_i$
$\quad\quad \triangleright$ Set line segment variable.
8 $\quad\quad \overline{AB} \leftarrow \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$
9 $\quad\quad \Theta_{\lfloor \ell_1 \rfloor + 1} \leftarrow \Phi_1(\overline{AB}, \lfloor \ell_1 \rfloor + 1, \Pi, \Delta)$
10 $\quad\quad \Theta_{\lceil \ell_2 \rceil - 1} \leftarrow \Phi_2(\overline{AB}, \lceil \ell_2 \rceil - 1, \Pi, \Delta)$
11 $\quad\quad \Theta_{\lfloor \ell_1 \rfloor} \leftarrow \Phi_3(\overline{AB}, \lfloor \ell_1 \rfloor, \Pi, \Delta)$
12 $\quad\quad \Theta_{\lceil \ell_2 \rceil} \leftarrow \Phi_3(\overline{AB}, \lceil \ell_2 \rceil, \Pi, \Delta)$
13 $\quad\quad \Theta_{t_a}^{\min} \leftarrow \min\{\Theta_{\lfloor \ell_1 \rfloor + 1}, \Theta_{\lceil \ell_2 \rceil - 1}, \Theta_{\lfloor \ell_1 \rfloor}, \Theta_{\lceil \ell_2 \rceil}\}$
14 $\quad\quad \Theta_i^{\min} \leftarrow \min\{\Theta_i^{\min}, \Theta_{t_a}^{\min}\}$
15 $\quad$ **end** (of inner loop)
16 $\quad \Theta^{\min} \leftarrow \max\{\Theta^{\min}, \Theta_i^{\min}\}$
17 **end** (of outer loop)
18 **return** $\Theta^{\min}$

Figure 3. Pseudo-code for determining minimum capacity for a periodic resource given Π, Δ, and τ using fixed-priority scheduling algorithm. Note the algorithm is exact when k equals ∞. See the description of Section 4 for definition of ℓ_1, ℓ_2, and the Φ functions.

$$\ell_1 \stackrel{\text{def}}{=} \frac{(t_{a+1} - \Delta) + \sqrt{(t_{a+1} - \Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi}, \tag{14}$$

$$\ell_2 \stackrel{\text{def}}{=} \frac{(t_a - \Delta) + \sqrt{(t_a - \Delta)^2 + 4\Pi \bar{D}_{t_a}}}{2\Pi}. \tag{15}$$

That is, we consider $\lfloor \ell_1 \rfloor$, $\lfloor \ell_1 \rfloor + 1$, $\lceil \ell_2 \rceil - 1$ and $\lceil \ell_2 \rceil$ to evaluate Θ_ℓ. The logic behind the choice of Φ functions and our definition of ℓ_1 and ℓ_2 will be more apparent in the proof of correctness section below.

Since we are looking for only one point in $t \in (0, d_i]$ for task τ_i where $\widehat{W}_i(t) \leq \mathsf{sbf}(\Omega, t)$, we only need a single line segment of $\widehat{W}_i(t)$ that intersects with $\mathsf{sbf}(\Omega, t)$ and gives minimum capacity. Thus, we set $\Theta_i^{\min}$ to be the minimum of all $\Theta_{t_a}^{\min}$ values for each of the line segment of $\widehat{W}_i$. Finally, we set $\Theta^{\min}$ to be the maximum of all $\Theta_i^{\min}$ values. This ensures that for each task $\tau_i \in \tau$, we find a $t \leq d_i$ such that $\widehat{W}_i(t) \leq \mathsf{sbf}((\Pi, \Theta^{\min}, \Delta), t)$. Since $\widehat{W}_i(t) \geq W_i(t)$ for all t, this implies Theorem 1; thus τ is fixed priority schedulable upon EDP resource $\Omega = (\Pi, \Theta^{\min}, \Delta)$.

§**Algorithm Complexity.** The complexity of FPMinimumCapacity depends on the number of tasks n in the task set τ and the cardinality of testing set $\widehat{S}_i(\tau, k)$ for each task τ_i. The outer loop of the algorithm (Lines 2 to 17) iterates for each task, thus n times in total. The inner loop (Lines 4 to 15) scans every pair of testing set points in $\widehat{S}_i(\tau, k)$ (in non-decreasing order) for task τ_i, and this can take at most $1 + (i-1)(k-1)$ times for a single task. Using a "heap-of-heaps" described by Mok [19], the time complexity to obtain an element of the testing set is $O(\log n)$. Setting $\bar{D}_{t_a}$, $D_{t_{a+1}}$ and α (Lines 5, 6 and 7) is done in constant time on each iteration of the inner loop. Again, setting ℓ values and evaluating Θ values using these (Line 9 to 12) takes constant time. Therefore, the runtime complexity of FPMinimumCapacity is $O(\log n \cdot \sum_{i=1}^{n} |\widehat{S}_i(\tau, k)|)$. If $k = \infty$, the complexity for exactly determining the minimum capacity is the same complexity as the test of Theorem 1 on a fixed Ω, which may be pseudo-polynomial depending on the period of tasks. Otherwise, if k is a fixed integer, the complexity is at most $O\left(\log n \cdot \sum_{i=1}^{n}(1 + (i-1)(k-1))\right)$ times, which is $O(kn^2 \log n)$.

§**Algorithm Correctness.** To prove the correctness of FPMinimumCapacity, we prove the following theorem which states that the value returned by the algorithm (i.e., $\Theta^{\min}$) is at least the optimal minimum capacity value $\Theta^*(\Pi, \Delta, \tau)$. Furthermore, if the input k equals ∞, then the returned capacity is optimal.

Theorem 2 *For all $k \in \mathbb{N}^+ \cup \{\infty\}$,* FPMINIMUMCAPACITY *returns $\Theta^{\min} \geq \Theta^*(\Pi, \Delta, \tau)$. Furthermore, if $k = \infty$, $\Theta^{\min} = \Theta^*(\Pi, \Delta, \tau)$.*

We require some additional definitions similar to [15] for notational convenience. The next definition quantifies the minimum capacity $\Theta(\leq \Delta)$ that is required for sbf to exceed the line segment $\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$ at some point (t, D_t). We will use the convention that $\inf$ returns ∞ on an empty set.

Definition 4 (Minimum Capacity for $\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$)

$$\begin{aligned} &\Theta^* \left(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle\right) \\ &\stackrel{\text{def}}{=} \inf \left\{ \Theta \in \mathbb{R}^+ \;\middle|\; \begin{array}{ll} & (\Theta \leq \Delta) \\ \wedge & (\exists (t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle : D_t \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t)) \end{array} \right\}. \end{aligned} \tag{16}$$

The next function determines the minimum capacity for any given line segment $\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$ to have a point in the ℓ-feasibility region.

Definition 5 (ℓ-Minimum Capacity for $\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$)

$$\begin{aligned} &\Theta^*_\ell \left(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle\right) \\ &\stackrel{\text{def}}{=} \inf \left\{ \Theta(\leq \Delta) \in \mathbb{R}^+ \;\middle|\; \begin{array}{l} \exists (t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle \\ : (t, D_t) \in \mathcal{F}_\ell(\Pi, \Theta, \Delta) \end{array} \right\}. \end{aligned} \tag{17}$$

Note the two above definitions use infimum, since they are defined over infinite sets; however, we will later see (Corollary 4) that the infimum corresponds to the minimum (i.e., the value returned by $\inf$ exists in the set specified in the right-hand side of Equations 16 and 17).

In order to prove Theorem 2, we must prove some additional lemmas. We start by presenting the three conditions on the value of Θ that are necessary and sufficient condition for a line segment $\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$ to have a point in the ℓ-feasibility region.

Lemma 3 *For any two consecutive pair of values $(t_a, t_{a+1}) \in \widehat{S}_i(\tau, k)$, there exists $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$ such that $(t, D_t) \in \mathcal{F}_\ell(\Pi, \Delta, \Theta)$ for some $\ell \in \mathbb{N}^+$, if and only if, the following conditions hold:*

$$\begin{array}{lll} & \Theta \geq \Phi_1(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell, \Pi, \Delta) & (18a) \\ \wedge & \Theta \geq \Phi_2(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell, \Pi, \Delta) & (18b) \\ \wedge & \Theta \geq \Phi_3(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell, \Pi, \Delta) & (18c) \end{array} \tag{18}$$

Proof: For the "only if" direction, we must show if some point of the line segment is in the ℓ-feasibility region for any given $\ell \in \mathbb{N}^+$ then the three conditions of Equation (18) hold. We will show this by contrapositive; that is, if any of the three conditions is violated, the line segment will not be in $\mathcal{F}_\ell(\Pi, \Delta, \Theta)$ for that ℓ. We now consider the negation of the conditions of Equation (18). By negation, at least one of the Equations (18a), (18b), or (18c) must be violated. We will show that if any of the conditions is violated, then for all $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$, $(t, D_t) \notin \mathcal{F}_\ell(\Pi, \Delta, \Theta)$.

Case 1: *Equation (18a) is false.* That is,

$$\begin{array}{lll} & \Theta & < \frac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell+1} \\ \Rightarrow & \Theta & < \frac{\bar{D}_{t_a} + \alpha(t_{a+1} - t_a) - t_{a+1} + \ell\Pi + \Delta}{\ell+1}. \end{array}$$

The second inequality follows from the fact that $D_{t_{a+1}} = \bar{D}_{t_a} + \alpha(t_{a+1} - t_a)$. Consider any $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$. Let $x \stackrel{\text{def}}{=} t - t_a$ where $0 \leq x \leq t_{a+1} - t_a$; thus, $t = t_a + x$ and $D_t = \bar{D}_{t_a} + \alpha x$. Consider the expression

$$\frac{(\bar{D}_{t_a} + \alpha x) - (t_a + x) + \ell\Pi + \Delta}{\ell + 1}$$

Obviously, the above expression is non-increasing in x, since $U(\tau) \leq 1$ and α is at most the utilization of tasks with higher priority than τ_i. Therefore, $\frac{\bar{D}_{t_a} + \alpha(t_{a+1} - t_a) - t_{a+1} + \ell\Pi + \Delta}{\ell+1} \leq \frac{(\bar{D}_{t_a} + \alpha x) - (t_a + x) + \ell\Pi + \Delta}{\ell+1} \leq \frac{D_t - t + \ell\Pi + \Delta}{\ell+1}$ for all $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$. This implies that the first condition of ℓ-feasibility is violated for all (t, D_t).

Case 2: *Equation (18b) is false*. That is, $\Theta < \bar{D}_{t_a}/\ell$. Again, consider any $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$. Observe that $D_t = \bar{D}_{t_a} + \alpha(t - t_a) \geq \bar{D}_{t_a}$, since $t \geq t_a$ and $\alpha \geq 0$. Thus, $\Theta < \bar{D}_{t_a}/\ell$ implies $\Theta < D_t/\ell$ for all (t, D_t); this implies that the second condition of $\mathcal{F}_\ell(\Pi, \Delta, \Theta)$ is violated.

Case 3: *Equation (18c) is false*. That is,

$$\Theta < \frac{\bar{D}_{t_a} + \alpha(\ell\Pi + \Delta - t_a)}{\ell + \alpha}. \tag{19}$$

Consider any $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$. We consider two further subcases based on the value of t. We will show in both subcases, $(t, D_t) \notin \mathcal{F}_\ell(\Pi, \Delta, \Theta)$.

Subcase 3a: $t < \frac{\bar{D}_{t_a} - \alpha t_a + \ell\Pi + \Delta - (\ell+1)\Theta}{1-\alpha}$.
By solving for Θ, we obtain

$$\begin{aligned} & \Theta < \tfrac{\bar{D}_{t_a} - (1-\alpha)t - \alpha t_a + \ell\Pi + \Delta}{\ell+1} \\ \Rightarrow \quad & \Theta < \tfrac{D_t - t + \ell\Pi + \Delta}{\ell+1} \end{aligned}$$

The implication follows from $D_t = \bar{D}_{t_a} + \alpha(t - t_a)$. The above inequality implies that the first condition of ℓ-feasibility is violated.

Subcase 3b: $t \geq \frac{\bar{D}_{t_a} - \alpha t_a + \ell\Pi + \Delta - (\ell+1)\Theta}{1-\alpha}$.
Again, solving for Θ,

$$\begin{aligned} & \Theta \geq \tfrac{\bar{D}_{t_a} - (1-\alpha)t - \alpha t_a + \ell\Pi + \Delta}{\ell+1} \\ \Rightarrow \quad & \Theta \geq \tfrac{D_t - t + \ell\Pi + \Delta}{\ell+1} \end{aligned} \tag{20}$$

Now consider the value of the first partial derivative of Φ_3 with respect to α; i.e., $\frac{\partial \Phi_3}{\partial \alpha}$ which is equal to

$$\frac{\ell(\ell\Pi + \Delta - t_a) - \bar{D}_{t_a}}{(\ell + \alpha)^2}.$$

Note the sign of the above partial derivative is independent of the value of α; therefore, either $\frac{\partial \Phi_3}{\partial \alpha} \leq 0$, or $\frac{\partial \Phi_3}{\partial \alpha} > 0$ for any $\alpha : 0 \leq \alpha \leq 1$; in other words, the sign remains constant for all α. If $\frac{\partial \Phi_3}{\partial \alpha} > 0$, then Φ_3 is maximized when α is as large as possible (i.e., α equals one). By Equation (19), this implies that $\Theta < \frac{\bar{D}_{t_a} + \ell\Pi + \Delta - t_a}{\ell+1}$ which is impossible due to Equation (20). Thus, $\frac{\partial \Phi_3}{\partial \alpha} \leq 0$ must be true. If the partial derivative is non-positive, then Φ_3 is maximized when α is as small as possible (i.e., α equals zero). By Equation (19), $\Theta < \frac{D_t}{\ell}$ which violates the second condition of ℓ-feasibility.

Thus, we have proved that if the line segment has a point in the ℓ-feasibility region, then the conditions in Equation (18) hold.

For the "if" direction, we need to show, if the conditions hold then there exists a point on the line segment that is included in the ℓ-feasibility region. Again, we will show by contrapositive; that is, if the line segment is completely outside the ℓ-feasibility region, then there is a condition of Equation (18) that is not satisfied. Assume that for all $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$ that $(t, D_t) \notin \mathcal{F}_\ell(\Pi, \Delta, \Theta)$. The previous statement implies that the first or the second condition of ℓ-feasibility must be violated for each (t, D_t). We now consider two cases based on the "location" of the left end point of the line segment $(t_a, \bar{D}_{t_a})$.

Case 1: *The second condition of ℓ-feasibility is violated for* $(t_a, \bar{D}_{t_a})$. In this case, $\Theta < \frac{\bar{D}_{t_a}}{\ell}$. Indeed, this violates the condition of Equation (18b).

Case 2: *The second condition of ℓ-feasibility is not violated for* $(t_a, \bar{D}_{t_a})$. In this case, $\Theta \geq \frac{\bar{D}_{t_a}}{\ell}$. We now consider two further subcases regarding the "location" of $(t_{a+1}, D_{t_{a+1}})$.

Subcase 2a: *The first condition of ℓ-feasibility is violated for the right end point of line segment* $(t_{a+1}, D_{t_{a+1}})$. In this case, $\Theta < \frac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell+1}$. This clearly violates the condition of Equation (18a).

Subcase 2b: *The first condition of ℓ-feasibility is not violated for* $(t_{a+1}, D_{t_{a+1}})$. In this subcase, $\frac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell+1} \leq \Theta$. Consider the function $\theta(t) \stackrel{\text{def}}{=} \frac{\bar{D}_{t_a} + \alpha(t - t_a) - t + \ell\Pi + \Delta}{\ell+1}$ for $t \in [t_a, t_{a+1}]$. Thus, by this subcase and $D_{t_{a+1}} = \bar{D}_{t_a} + \alpha(t_{a+1} - t_a)$ we obtain the following equation,

$$\left(\theta(t_{a+1}) \stackrel{\text{def}}{=} \frac{\bar{D}_{t_a} + \alpha(t_{a+1} - t_a) - t_{a+1} + \ell\Pi + \Delta}{\ell + 1}\right) \leq \Theta. \tag{21}$$

By Case 2, the second condition of ℓ-feasibility is not violated for $(t_a, \bar{D}_{t_a})$. Thus, the first condition must be; i.e.,

$$\left(\theta(t_a) \stackrel{\text{def}}{=} \frac{\bar{D}_{t_a} - t_a + \ell\Pi + \Delta}{\ell + 1}\right) > \Theta. \tag{22}$$

Therefore, $\Theta \in [\theta(t_{a+1}), \theta(t_a))$. Observe that $\theta(t)$ is continuous for all $t \in [t_a, t_{a+1}]$. Therefore, the Intermediate Value Theorem implies that there exists a $t' \in [t_a, t_{a+1}]$ such that $\theta(t')$ equals Θ. That is,

$$\frac{\bar{D}_{t_a} + \alpha(t' - t_a) - t' + \ell\Pi + \Delta}{\ell + 1} = \Theta. \tag{23}$$

By the above equality, the first condition of ℓ-feasibility is not violated for $(t', D_{t'})$; therefore, the second condition must be false:

$$\frac{\bar{D}_{t_a} + \alpha(t' - t_a)}{\ell} > \Theta. \tag{24}$$

Solving Equation (23) for t', we obtain

$$t' = \frac{\bar{D}_{t_a} - \alpha t_a + \ell\Pi + \Delta - (\ell + 1)\Theta}{1 - \alpha}.$$

Substituting the above solution to t' into Equation (24) and solving for Θ, we obtain

$$\Theta < \frac{\bar{D}_{t_a} - \alpha(\ell\Pi + \Delta - t_a)}{\ell + \alpha}$$

which indeed violates the condition of Equation (18c).

Thus, if the line segment is strictly above the ℓ-feasibility region, at least one of the three conditions is violated. ■

The following lemma formalizes the equivalence between the concept of a line segment $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ being included in some ℓ-feasibility region and the concept of a cumulative request-bound function $\widehat{W}_i$ falling below a supply-bound function sbf.

Lemma 4 *For consecutive pair of values* $(t_a, t_{a+1}) \in \widehat{S}_i(\tau, k)$ *and* $(t, D_t) \in \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ *such that* $t_a < t \leq t_{a+1}$*, the inequality* $\widehat{W}_i(t) \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t)$ *holds, if and only if, there exists* $\ell \in \mathbb{N}^+$ *such that* $(t, D_t) \in \mathcal{F}_\ell(\Pi, \Delta, \Theta)$.

Proof Sketch: For the "if" direction, we must show that if the point $(t, D_t) \in \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ satisfies $(t, D_t) \in \mathcal{F}_\ell(\Pi, \Delta, \Theta)$, then there is sufficient supply over an interval of length t to satisfy the execution of a job of τ_i and the approximated execution times of all higher-priority tasks (formally, $\widehat{W}_i(t) \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t)$). Observe that every point in $\mathcal{F}_\ell(\Pi, \Delta, \Theta)$ is below the sbf function (see Figure 2)[1]. Thus, if $(t, D_t) \in \mathcal{F}_\ell(\Pi, \Delta, \Theta)$, then $D_t \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t)$. Finally, Lemma 2 states that $\widehat{W}_i(t) \leq D_t$ implying the "if" direction.

For the "only if" direction, observe that $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ and $\widehat{W}_i(t)$ are equivalent for $t \in (t_a, t_{a+1}]$. Thus, we must show that if line segment $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ has point (t, D_t) contained below the sbf function for Ω, then there exists an $\ell \in \mathbb{N}^+$ such that $\langle(t, D_t), \alpha\rangle \in \mathcal{F}_\ell(\Pi, \Theta, \Delta)$. Consider $\ell = \lceil \frac{D_t}{\Theta} \rceil$. The second condition of ℓ-feasibility (Equation (11)) is trivially satisfied for this ℓ. It also must be true that $D_t > (\ell - 1)\Theta$. Thus, (t, D_t) must be below of the line defined by $y = x - (\ell\Pi + \Delta - (\ell + 1)\Theta)$ (otherwise, (t, D_t) would be above the sbf function at t). This last constraint is equivalent to the first condition of ℓ-feasibility region. Therefore, for $\ell = \lceil \frac{D_t}{\Theta} \rceil$ we have satisfied the two conditions of Equation (11), implying that $(t, D_t) \in \mathcal{F}_{\lceil \frac{D_t}{\Theta} \rceil}(\Pi, \Theta, \Delta)$. ■

[1] A full algebraic proof of this is rather involved and will be included in an extended version of this paper

In the above lemma, we did not include t_a in the interval of time values where line segment inclusion in the ℓ-feasibility region implies that the approximate request-bound function is below the supply-bound function. The exclusion of t_a from the above lemma is due to the fact that $\widehat{W}_i$ is discontinuous at t_a. However, notice that t_a is the right end point of the predecessor line segment immediately to the left of $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$.

Lemma 3 equates the concept of finding t such that $\widehat{W}_i(t)$ is below the sbf for a given Θ and the concept of point (t, D_t) of a line segment $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ being contained in some ℓ-feasibility region for Θ. The next lemma uses Definitions 4 and 5 to show that if we can compute $\Theta^*_\ell(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$ for any $\ell \in \mathbb{N}^+$, then we can also compute $\Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$.

Lemma 5

$$\Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle) = \inf_{\ell > 0} \left\{\Theta^*_\ell(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)\right\}. \tag{25}$$

Proof: Let Θ_{RHS} denote the right-hand side of Equation (25). We will show that both $\Theta_{\text{RHS}} \geq \Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$ and $\Theta_{\text{RHS}} \leq \Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$ which will imply the lemma. First, we show $\Theta_{\text{RHS}} \geq \Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$. By definition of infimum, for any $\delta > 0$, there exists $\ell \in \mathbb{N}^+$ such that $\Theta^*_\ell(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle) \leq \Theta_{\text{RHS}} + \delta$. Definition 5 states that there exists $(t, D_t) \in \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ such that $(t, D_t) \in \mathcal{F}_\ell(\Pi, \Theta_{\text{RHS}} + \delta, \Delta)$ for this ℓ. Therefore, for all $\delta > 0$, $\Theta_{\text{RHS}} + \delta$ must be in the set considered in the inf on the right-hand side of Equation (16) by Definition 4. Thus, $\Theta_{\text{RHS}} \geq \Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$.

Next, we will show $\Theta_{\text{RHS}} \leq \Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$. By Definition 4 and application of Lemma 2, there exist $(t, D_t) \in \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$ such that

$$\widehat{W}_i(t) \leq \mathsf{sbf}((\Pi, \Delta, \Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)), t).$$

Lemma 4 implies that there exists $\ell \in \mathbb{N}^+$ such that $(t, D_t) \in \mathcal{F}_\ell(\Pi, \Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle), \Delta)$. By Definition 5, this implies that $\Theta^*(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$ is in the set considered in the right-hand side of Equation (17) which implies the inequality. ■

In the next few lemmas, we derive the values ℓ_1 and ℓ_2 (Equations (14) and (15)), and prove that we only need to evaluate the Φ functions at these ℓ values to obtain minimum capacity. Consider the three conditions given in Equation (18) of Lemma 3. There are three possible cases. We invite the reader to verify that these cases are complete and mutually exclusive. In the cases let $\overline{AB}$ denote the line segment $\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$.

Case I: $\left(\Phi_1(\overline{AB}, \ell, \Pi, \Delta) > \Phi_2(\overline{AB}, \ell, \Pi, \Delta)\right) \wedge \left(\Phi_1(\overline{AB}, \ell, \Pi, \Delta) > \Phi_3(\overline{AB}, \ell, \Pi, \Delta)\right)$;

Case II: $\left(\Phi_2(\overline{AB}, \ell, \Pi, \Delta) > \Phi_3(\overline{AB}, \ell, \Pi, \Delta)\right) \wedge \left(\Phi_2(\overline{AB}, \ell, \Pi, \Delta) \geq \Phi_1(\overline{AB}, \ell, \Pi, \Delta)\right)$;

Case III: $\left(\Phi_3(\overline{AB}, \ell, \Pi, \Delta) \geq \Phi_1(\overline{AB}, \ell, \Pi, \Delta)\right) \wedge \left(\Phi_3(\overline{AB}, \ell, \Pi, \Delta) \geq \Phi_2(\overline{AB}, \ell, \Pi, \Delta)\right)$.

For each of the above cases, we solve for the value of ℓ and obtain bounds for the value of ℓ.

Lemma 6 *For any $\overline{AB} \stackrel{\text{def}}{=} \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle$, $\ell \in \mathbb{N}^+$, Π, Δ, Case I holds, if and only if,*

$$\ell \geq \left\lfloor \frac{(t_{a+1} - \Delta) + \sqrt{(t_{a+1} - \Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi} \right\rfloor + 1. \tag{26}$$

Proof: Let us consider the "only if" direction of the lemma; that is, Case I holds. From Case I, we have that both $\Phi_1(\overline{AB}, \ell, \Pi, \Delta) > \Phi_2(\overline{AB}, \ell, \Pi, \Delta)$ and $\Phi_1(\overline{AB}, \ell, \Pi, \Delta) > \Phi_3(\overline{AB}, \ell, \Pi, \Delta)$. For $\Phi_1(\overline{AB}, \ell, \Pi, \Delta) > \Phi_2(\overline{AB}, \ell, \Pi, \Delta)$, solving for ℓ,

$$\begin{aligned} & \frac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell + 1} > \frac{\bar{D}_{t_a}}{\ell} \\ \Leftrightarrow \quad & \ell > \frac{[(1-\alpha)t_{a+1} + \alpha t_a - \Delta] + \sqrt{((1-\alpha)t_{a+1} + \alpha t_a - \Delta)^2 + 4\Pi \bar{D}_{t_a}}}{2\Pi}. \end{aligned} \tag{27}$$

The bidirectional implication follows since Inequality (27) is a quadratic inequality with respect to ℓ, defining a convex parabola $\Pi\ell^2 - ((1-\alpha)t_{a+1} + \alpha t_a - \Delta)\,\ell - \bar{D}_{t_a}$. The zeros of the parabola are

$$\frac{[(1-\alpha)t_{a+1} + \alpha t_a - \Delta] \pm \sqrt{((1-\alpha)t_{a+1} + \alpha t_a - \Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi}.$$

Since the square-root term in the numerator is always greater than the term preceding the $\pm$, one root is positive and the other is negative. Inequality (27) implies that we are interested in values of $\ell \in \mathbb{N}^+$ such that the parabola strictly exceeds zero. Since the parabola is convex, all values of ℓ strictly greater than the positive root satisfy this inequality.

For $\Phi_1(\overline{AB}, \ell, \Pi, \Delta) > \Phi_3(\overline{AB}, \ell, \Pi, \Delta)$, solving for ℓ,

$$\begin{aligned} & \tfrac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell+1} > \tfrac{\bar{D}_{t_a} + \alpha(\ell\Pi + \Delta - t_a)}{\ell+\alpha} \\ \Leftrightarrow \quad & \ell > \tfrac{(t_{a+1}-\Delta) + \sqrt{(t_{a+1}-\Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi} \end{aligned} \tag{28}$$

The bidirectional implication follows since Inequality (28) is a quadratic inequality with respect to ℓ, defining a convex parabola $\Pi\ell^2 - (t_{a+1} - \Delta)\,\ell - \left(\left(\bar{D}_{t_a} + \alpha(t_{a+1} - t_a)\right)\right)$. By similar reasoning done for Inequality (27), all values of ℓ strictly greater than the positive root satisfy this inequality.

Combining Equations (27) and (28), we obtain

$$\ell > \max \left\{ \begin{array}{l} \frac{[(1-\alpha)t_{a+1} + \alpha t_a - \Delta] + \sqrt{((1-\alpha)t_{a+1} + \alpha t_a - \Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi}, \\ \frac{(t_{a+1}-\Delta) + \sqrt{(t_{a+1}-\Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi} \end{array} \right\}. \tag{29}$$

Observe that $(1-\alpha)t_{a+1} + \alpha t_a - \Delta$ equals $t_{a+1} - \Delta - \alpha(t_{a+1} - t_a)$ which is at most $t_{a+1} - \Delta$, since $t_{a+1} > t_a$ and $0 \le \alpha \le 1$. Thus, we conclude that the second value of Equation (29) is the maximum of the two bounds obtained in this case. The lemma follows by observing that ℓ is an integer. The "if" direction follows by simply reversing the direction of each implication in the proof. ■

Lemma 7 *For any* $\overline{AB} \overset{def}{=} \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$, $\ell \in \mathbb{N}^+$, Π, Δ, *Case II holds, if and only if,*

$$\ell \le \left\lceil \frac{(t_a - \Delta) + \sqrt{(t_a - \Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi} \right\rceil - 1. \tag{30}$$

Proof: Let us consider the "only if" direction of the lemma; that is, Case II holds. From Case II, we have that both $\Phi_2(\overline{AB}, \ell, \Pi, \Delta) > \Phi_3(\overline{AB}, \ell, \Pi, \Delta)$ and $\Phi_2(\overline{AB}, \ell, \Pi, \Delta) \ge \Phi_1(\overline{AB}, \ell, \Pi, \Delta)$. For $\Phi_2(\overline{AB}, \ell, \Pi, \Delta) > \Phi_3(\overline{AB}, \ell, \Pi, \Delta)$, solving for ℓ,

$$\begin{aligned} & \tfrac{\bar{D}_{t_a}}{\ell} > \tfrac{\bar{D}_{t_a} + \alpha(\ell\Pi + \Delta - t_a)}{\ell+\alpha} \\ \Leftrightarrow \quad & \ell < \tfrac{(t_a-\Delta) + \sqrt{(t_a-\Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi} \end{aligned} \tag{31}$$

The bidirectional implication follows since Inequality (31) is a quadratic inequality with respect to ℓ, defining a convex parabola $\Pi\ell^2 - (t_a - \Delta)\,\ell - \bar{D}_{t_a}$. The zeros of the parabola are

$$\frac{(t_a - \Delta) \pm \sqrt{(t_a - \Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi}.$$

Since the square-root term in the numerator is always greater than the term preceding the $\pm$, one root is positive and the other is negative. Inequality (31) implies that we are interested in values of $\ell \in \mathbb{N}^+$ such that the parabola is strictly below zero. Since the parabola is convex, all positive integer values of ℓ strictly less than the positive root satisfy this inequality.

For $\Phi_2(\overline{AB}, \ell, \Pi, \Delta) \ge \Phi_1(\overline{AB}, \ell, \Pi, \Delta)$, solving for ℓ,

$$\begin{aligned} & \tfrac{\bar{D}_{t_a}}{\ell} \ge \tfrac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell+1} \\ \Leftrightarrow \quad & \ell \le \tfrac{[(1-\alpha)t_{a+1} + \alpha t_a - \Delta] + \sqrt{((1-\alpha)t_{a+1} + \alpha t_a - \Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi} \end{aligned} \tag{32}$$

The bidirectional implication follows since Inequality (32) is a quadratic inequality with respect to ℓ, defining a convex parabola $\Pi\ell^2 - ((1-\alpha)t_{a+1} + \alpha t_a - \Delta)\,\ell - \bar{D}_{t_a}$. By similar reasoning done for Inequality (31), all positive integer values of ℓ at most the positive root satisfy this inequality.

Now consider the following term: $(1-\alpha)t_{a+1} + \alpha t_a - \Delta$ which equals $(1-\alpha)(t_{a+1} - t_a) + t_a - \Delta$ which is at least $t_a - \Delta$ since $t_{a+1} > t_a$ and $0 \le \alpha \le 1$. Thus, we conclude that the value on the right-hand-side of Equation (31) is the minimum of the two values obtained in this case. The lemma follows by observing that ℓ must be an integer. The "if" direction of the lemma follows by simply reversing the implications of the proof. ■

Lemma 8 *For any* $\overline{AB} \stackrel{def}{=} \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$, $\ell \in \mathbb{N}^+$, Π, Δ, *Case III holds, if and only if,*

$$\left\lceil \frac{(t_a - \Delta) + \sqrt{(t_a - \Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi} \right\rceil \le \ell \le \left\lfloor \frac{(t_{a+1} - \Delta) + \sqrt{(t_{a+1} - \Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi} \right\rfloor \tag{33}$$

Proof: Let us consider the "only if" direction of the lemma; that is, Case III holds. From Case III, we have that both $\Phi_3(\overline{AB}, \ell, \Pi, \Delta) \ge \Phi_1(\overline{AB}, \ell, \Pi, \Delta)$ and $\Phi_3(\overline{AB}, \ell, \Pi, \Delta) \ge \Phi_2(\overline{AB}, \ell, \Pi, \Delta)$. For $\Phi_3(\overline{AB}, \ell, \Pi, \Delta) \ge \Phi_1(\overline{AB}, \ell, \Pi, \Delta)$, solving for ℓ,

$$\begin{aligned} &\tfrac{\bar{D}_{t_a} + \alpha(\ell\Pi + \Delta - t_a)}{\ell + \alpha} \ge \tfrac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell + 1} \\ \Leftrightarrow\ &\ell \le \tfrac{(t_{a+1} - \Delta) + \sqrt{(t_{a+1} - \Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi} \end{aligned} \tag{34}$$

The bidirectional implication follows since Inequality (34) is a quadratic inequality with respect to ℓ, defining a convex parabola $\Pi\ell^2 - (t_{a+1} - \Delta)\,\ell - \left(\bar{D}_{t_a} + \alpha(t_{a+1} - t_a)\right)$. The zeros of the parabola are

$$\frac{(t_{a+1} - \Delta) \pm \sqrt{(t_{a+1} - \Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi}.$$

Since the square-root term in the numerator is always greater than the term preceding the $\pm$, one root is positive and the other is negative. Inequality (34) implies that we are interested in values of $\ell \in \mathbb{N}^+$ such that the parabola is at most zero. Since the parabola is convex, all positive integer values of ℓ at most the positive root satisfy this inequality.

For $\Phi_3(\overline{AB}, \ell, \Pi, \Delta) \ge \Phi_2(\overline{AB}, \ell, \Pi, \Delta)$, solving for ℓ,

$$\begin{aligned} &\tfrac{\bar{D}_{t_a} + \alpha(\ell\Pi + \Delta - t_a)}{\ell + \alpha} \ge \tfrac{\bar{D}_{t_a}}{\ell} \\ \Leftrightarrow\ &\ell \ge \tfrac{(t_a - \Delta) + \sqrt{(t_a - \Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi} \end{aligned} \tag{35}$$

The bidirectional implication follows since Inequality (35) is a quadratic inequality with respect to ℓ, defining a convex parabola $\Pi\ell^2 - (t_a - \Delta)\,\ell - \bar{D}_{t_a}$. The zeros of the parabola are

$$\ell \ge \frac{(t_a - \Delta) \pm \sqrt{(t_a - \Delta)^2 + 4\Pi\bar{D}_{t_a}}}{2\Pi}.$$

Since the square-root term in the numerator is always greater than the term preceding the $\pm$, one root is positive and the other is negative. Inequality (35) implies that we are interested in values of $\ell \in \mathbb{N}^+$ such that the parabola is at least zero. Since the parabola is convex, all positive integer values of ℓ at least the positive root satisfy this inequality.

The lemma follows by observing that ℓ must be an integer. The "if" direction of the lemma follows by simply reversing the implications of the proof.

■

We now prove three lemmas and corollaries which show that for all $\ell \in \mathbb{N}^+$ not equal to the values $\lfloor \ell_1 \rfloor$, $\lfloor \ell_1 \rfloor + 1$, $\lceil \ell_2 \rceil$ or $\lceil \ell_2 \rceil - 1$ will result in a larger minimum Θ. The first lemma, towards this goal, shows that if a point on the line segment is in an ℓ'-feasibility region and ℓ' is at least $\lfloor \ell_1 \rfloor + 1$, then the point is also in the $\lfloor \ell_1 \rfloor + 1$-feasibility region.

Lemma 9 *For any* $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$, $\ell' \in \mathbb{N}^+$, Π, Δ, *and* Θ, *if* $\ell' \ge \lfloor \ell_1 \rfloor + 1$ *and* $\Theta \le \Delta$ *then*

$$[(t, D_t) \in \mathcal{F}_{\ell'}(\Pi, \Delta, \Theta)] \Rightarrow \left[(t, D_t) \in \mathcal{F}_{\lfloor \ell_1 \rfloor + 1}(\Pi, \Delta, \Theta)\right].$$

Proof: By Lemma 6 and $\ell' \geq \lfloor \ell_1 \rfloor + 1$, Case I must hold for all such ℓ'. Combining Case I and Lemma 3, we have that if $(t, D_t) \in \mathcal{F}_{\ell'}(\Pi, \Delta, \Theta)$, then

$$\Theta \geq \Phi_1(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell', \Pi, \Delta).$$

Now consider the first partial derivative of Φ_1 with respect to ℓ; i.e.,

$$\begin{aligned} \frac{\partial \Phi_1}{\partial \ell} &= \frac{-D_{t_{a+1}} + t_{a+1} - \ell\Pi - \Delta + \Pi(\ell+1)}{(\ell+1)^2} \\ &= \frac{[t_{a+1} - D_{t_{a+1}}] + [\Pi - \Delta]}{(\ell+1)^2}. \end{aligned}$$

Since $\Pi \geq \Delta$, the second term in the numerator is positive. Consider the first term, $t_{a+1} - D_{t_{a+1}}$. By $(t, D_t) \in \mathcal{F}_{\ell'}(\Pi, \Delta, \Theta)$ and the first condition of ℓ'-feasibility,

$$\begin{aligned} & t \geq D_t + \ell\Pi + \Delta - (\ell+1)\Theta \\ \Rightarrow \quad & t + (t_{a+1} - t) \geq D_t + \alpha(t_{a+1} - t) + \ell\Pi + \Delta - (\ell+1)\Theta \\ & \qquad \text{(since } \alpha < 1\text{)} \\ \Rightarrow \quad & t_{a+1} \geq D_{t_{a+1}} + \ell\Pi + \Delta - (\ell+1)\Theta \\ \Rightarrow \quad & t_{a+1} \geq D_{t_{a+1}}. \end{aligned}$$

The second to last implication is due to $D_t = \bar{D}_{t_a} + \alpha(t - t_a)$ and $D_{t_{a+1}} = \bar{D}_{t_a} + \alpha(t_{a+1} - t_a)$. The last implication is due to $\Theta \leq \Delta$. Therefore, the first term in the numerator of $\frac{\partial \Phi_1}{\partial \ell}$ is also positive. Thus, $\frac{\partial \Phi_1}{\partial \ell}$ is non-decreasing for all ℓ'. Thus, the Φ_1 evaluated at $\lfloor \ell_1 \rfloor + 1$ is a lower bound; i.e., for all $\ell' \geq \lfloor \ell_1 \rfloor + 1$,

$$\Phi_1(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell', \Pi, \Delta) \geq \Phi_1(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \lfloor \ell_1 \rfloor + 1, \Pi, \Delta).$$

The above inequality implies that $\Theta \geq \Phi_1(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \lfloor \ell_1 \rfloor + 1, \Pi, \Delta)$, satisfying Equation (18a) of Lemma 3. For $\lfloor \ell_1 \rfloor + 1$, Case I holds, implying that Equations (18b) and (18c) must also hold. Thus, by Lemma 3, $(t, D_t) \in \mathcal{F}_{\lfloor \ell_1 \rfloor + 1}(\Pi, \Delta, \Theta)$. ■

The next corollary follows from the above lemma and the definition of Θ^*_ℓ (Definition 5).

Corollary 1 *For any $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, $\ell' \in \mathbb{N}^+$, Π, and Δ, if $(\ell' \geq \lfloor \ell_1 \rfloor + 1)$ then*

$$\Theta^*_{\ell'}(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle) \geq \Theta^*_{\lfloor \ell_1 \rfloor + 1}(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle).$$

The next lemma shows that if a point on the line segment is in an ℓ'-feasibility region and ℓ' is at most $\lceil \ell_2 \rceil - 1$, then the point is also in the $\lceil \ell_2 \rceil - 1$-feasibility region.

Lemma 10 *For any $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$, $\ell' \in \mathbb{N}^+$, Π, Δ, and Θ, if $\ell' \leq \lceil \ell_2 \rceil - 1$ and $\Theta \leq \Delta$ then*

$$[(t, D_t) \in \mathcal{F}_{\ell'}(\Pi, \Delta, \Theta)] \Rightarrow \left[(t, D_t) \in \mathcal{F}_{\lceil \ell_2 \rceil - 1}(\Pi, \Delta, \Theta)\right].$$

Proof:

By Lemma 7 and $\ell' \leq \lceil \ell_2 \rceil - 1$, Case II must hold for all such ℓ'. Combining Case II and Lemma 3, we have that if $(t, D_t) \in \mathcal{F}_{\ell'}(\Pi, \Delta, \Theta)$, then

$$\Theta \geq \Phi_2(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell', \Pi, \Delta).$$

Now consider the first partial derivative of Φ_2 with respect to ℓ;

$$\frac{\partial \Phi_2}{\partial \ell} = \frac{-D_t}{\ell^2}.$$

Therefore, Φ_2 is a decreasing function for all $\ell' \in \mathbb{N}^+$ such that $\ell' \leq \lceil \ell_2 \rceil - 1$. Thus, the Φ_2 evaluated at $\lceil \ell_2 \rceil - 1$ is an upper bound for all such ℓ'; i.e., for all $\ell' \leq \lceil \ell_2 \rceil - 1$,

$$\Phi_2(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell', \Pi, \Delta) \leq \Phi_2(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \lceil \ell_2 \rceil - 1, \Pi, \Delta).$$

The above inequality implies that $\Theta \geq \Phi_2(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \lceil \ell_2 \rceil - 1, \Pi, \Delta)$, satisfying Equation (18b) of Lemma 3. For $\lceil \ell_2 \rceil - 1$, Case II holds, implying that Equations (18a) and (18c) must also hold. Thus, by Lemma 3, $(t, D_t) \in \mathcal{F}_{\lceil \ell_2 \rceil - 1}(\Pi, \Delta, \Theta)$.

■

The next corollary follows from the above lemma and the definition of Θ^*_ℓ (Definition 5).

Corollary 2 *For any* $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, $\ell' \in \mathbb{N}^+$, Π, *and* Δ, *if* $(\ell' \leq \lceil \ell_2 \rceil - 1)$ *then*

$$\Theta^*_{\ell'}(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle) \geq \Theta^*_{\lceil \ell_2 \rceil - 1}(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle).$$

Lemma 11 *For any* $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$, $\ell' \in \mathbb{N}^+$, Π, Δ, *and* Θ, *if* $\lceil \ell_2 \rceil \leq \ell' \leq \lfloor \ell_1 \rfloor$ *then*

$$[(t, D_t) \in \mathcal{F}_{\ell'}(\Pi, \Delta, \Theta)] \Rightarrow \left[(t, D_t) \in \mathcal{F}_{\lfloor \ell_1 \rfloor}(\Pi, \Delta, \Theta)\right] \vee \left[(t, D_t) \in \mathcal{F}_{\lceil \ell_2 \rceil}(\Pi, \Delta, \Theta)\right].$$

Proof: By Lemma 8 and $\lceil \ell_2 \rceil \leq \ell' \leq \lfloor \ell_1 \rfloor$, Case III must hold for all such ℓ'. Combining Case III and Lemma 3, we have that if $(t, D_t) \in \mathcal{F}_{\ell'}(\Pi, \Delta, \Theta)$, then

$$\Theta \geq \Phi_3(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell', \Pi, \Delta).$$

Now consider the first partial derivative of Φ_3 with respect to ℓ;

$$\frac{\partial \Phi_3}{\partial \ell} = \frac{\alpha^2 \Pi - \bar{D}_{t_a} - \alpha \Delta + \alpha t_a}{(\ell + \alpha)^2}.$$

Note the sign of the above partial derivative is independent of the value of ℓ; therefore, either $\frac{\partial \Phi_3}{\partial \ell} \leq 0$, or $\frac{\partial \Phi_3}{\partial \ell} > 0$ for any $\ell \in \mathbb{N}^+$; in other words, the sign remains constant for all ℓ. If $\frac{\partial \Phi_3}{\partial \ell} > 0$, then Φ_3 is minimized when ℓ is as small as possible; i.e., ℓ equals $\lceil \ell_2 \rceil$. In this case, the Φ_3 evaluated at $\lceil \ell_2 \rceil$ is a lower bound for all such ℓ'; i.e., for all ℓ' such that $\lceil \ell_2 \rceil \leq \ell' \leq \lfloor \ell_1 \rfloor$,

$$\Phi_3(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell', \Pi, \Delta) \geq \Phi_3(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \lceil \ell_2 \rceil, \Pi, \Delta).$$

The above inequality implies that $\Theta \geq \Phi_3(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \lceil \ell_2 \rceil, \Pi, \Delta)$, satisfying Equation (18c) of Lemma 3. For $\lceil \ell_2 \rceil$, Case III holds, implying that Equations (18a) and (18b) must also hold. Thus, by Lemma 3, $(t, D_t) \in \mathcal{F}_{\lceil \ell_2 \rceil}(\Pi, \Delta, \Theta)$ when $\frac{\partial \Phi_3}{\partial \ell} > 0$.

If $\frac{\partial \Phi_3}{\partial \ell} \leq 0$, then Φ_3 is minimized when ℓ is as large as possible; i.e., ℓ equals $\lfloor \ell_1 \rfloor$. In this case, the Φ_3 evaluated at $\lfloor \ell_1 \rfloor$ is an upper bound for all such ℓ'; i.e., for all ℓ' such that $\lceil \ell_2 \rceil \leq \ell' \leq \lfloor \ell_1 \rfloor$,

$$\Phi_3(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell', \Pi, \Delta) \geq \Phi_3(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \lfloor \ell_1 \rfloor, \Pi, \Delta).$$

By the same argument for $\frac{\partial \Phi_3}{\partial \ell} > 0$, $(t, D_t) \in \mathcal{F}_{\lfloor \ell_1 \rfloor}(\Pi, \Delta, \Theta)$ when $\frac{\partial \Phi_3}{\partial \ell} \leq 0$. ■

The next corollary follows from the above lemma and the definition of Θ^*_ℓ (Definition 5).

Corollary 3 *For any* $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, $\ell' \in \mathbb{N}^+$, Π, *and* Δ, *if* $(\lceil \ell_2 \rceil \leq \ell' \leq \lfloor \ell_1 \rfloor)$ *then*

$$\Theta^*_{\ell'}(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle) \leq \min\{\Theta^*_{\lfloor \ell_1 \rfloor}(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle), \Theta^*_{\lceil \ell_2 \rceil}(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)\}.$$

Combining Corollaries 1, 2, 3, and using Definitions 4 and 5, we obtain the following corollary.

Corollary 4

$$\Theta^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle) = \min_{\ell \in \{\lfloor \ell_1 \rfloor, \lfloor \ell_1 \rfloor + 1, \lceil \ell_2 \rceil, \lceil \ell_2 \rceil - 1\}} \{\Theta^*_\ell(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)\}.$$

Proof: By Lemma 5, we may determine $\Theta^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)$ by evaluating $\Theta^*_\ell(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)$ for all possible $\ell \in \mathbb{N}^+$. The corollary follows by applying Corollaries 1, 2, and 3, respectively, for the following regions of ℓ: $[1, \lceil \ell_2 \rceil - 1]$, $[\lceil \ell_2 \rceil, \lfloor \ell_1 \rfloor]$, and $[\lfloor \ell_1 \rfloor + 1, \infty)$. ■

By the above corollary, we now know how to compute $\Theta^*(\cdot)$ efficiently from $\Theta^*_\ell(\cdot)$. The next lemma shows that we may use the Φ functions to efficiently compute $\Theta^*_\ell(\cdot)$.

Lemma 12 *Let* $\overline{AB}$ *represent* $\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$. *For any* $\ell \in \mathbb{N}^+$,

$$\Theta^*_\ell(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle) = \begin{cases} \Phi_1(\overline{AB}, \ell, \Pi, \Delta), & \text{if } \ell \geq \lfloor \ell_1 \rfloor + 1; \\ \Phi_2(\overline{AB}, \ell, \Pi, \Delta), & \text{if } \ell \leq \lceil \ell_2 \rceil - 1; \\ \Phi_3(\overline{AB}, \ell, \Pi, \Delta), & \text{otherwise.} \end{cases} \quad (36)$$

Proof: From Definition 5, $\Theta^*_\ell(\Pi, \Delta, \overline{AB})$ is the minimum $\Theta \leq \Delta$ such that there exists $(t, D_t) \in \overline{AB}$ where $(t, D_t) \in \mathcal{F}_\ell(\Pi, \Theta, \Delta)$. By Lemma 3, such a Θ is also the minimum value that satisfied all three conditions of Equation (18). Since each of the conditions is a lower bound on Θ (with equality permitted), Θ must satisfy equality of at least one of the three conditions of Equation (18) and must exceed or equal the other two conditions. Notice that, if $\ell \geq \lfloor \ell_1 \rfloor + 1$, then by Lemma 6, Θ equals $\Phi_1(\overline{AB}, \ell, \Pi, \Delta)$. We can show an identical proof for intervals $(0, \lceil \ell_2 \rceil - 1]$ and$[\lceil \ell_2 \rceil, \lfloor \ell_1 \rfloor]$, by applying Lemmas 7 and 8, respectively. ■

The final lemma that we prove before providing a proof for Theorem 2 shows that a choice of Θ based on the computation of $\Theta^*(\cdot)$ is a "safe" choice in the sense that all tasks in τ_i will complete by their deadline under an EDP resource $\Omega = (\Pi, \Theta, \Delta)$.

Lemma 13 *For all $\tau_i \in \tau$, $\exists t \in (0, d_i]$ such that $\widehat{W}_i(t) \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t)$ and $U(\tau) \leq \frac{\Theta}{\Pi}$, if and only if,*

$$\Theta \geq \max \left(\begin{array}{l} \max_{\tau_i \in \tau} \left\{ \min_{t_a, t_{a+1} \in \widehat{S}_i} \left\{ \Theta^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle) \right\} \right\}, \\ U(\tau) \cdot \Pi \end{array} \right). \quad (37)$$

Proof: We will prove this lemma by contrapositive. For the "if" direction, we must prove if either $U(\tau) > \frac{\Theta}{\Pi}$ or $\forall t \in (0, d_i] : \widehat{W}_i(t) > \mathsf{sbf}((\Pi, \Theta, \Delta), t)$, then the negation of the inequality of Equation (37) is true. If we consider $U(\tau) > \frac{\Theta}{\Pi}$, the inequality of Equation (37) is trivially violated due to the second expression in the outer $\max$ of Equation (37).

Now, consider the case when there exists a $\tau_i \in \tau$ such that $\widehat{W}_i(t) > \mathsf{sbf}((\Pi, \Theta, \Delta), t)$ for all t in $(0, d_i]$. By Lemma 4, this implies for all $\ell \in \mathbb{N}^+$, $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, and $(t, D_t) \in \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle$ that $(t, D_t) \notin \mathcal{F}_\ell(\Pi, \Delta, \Theta)$. By Definition 5, it must be for all $\ell \in \mathbb{N}^+$ that $\Theta < \Theta^*_\ell(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)$. By Lemma 5, this implies that $\Theta < \Theta^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)$ for any $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, which violates the inequality of Equation (37) due to the first term in the outer $\max$. For the "only if" direction of the lemma, we will also consider the contrapositive. The contrapositive will follow by simply reversing the implications of the proof for the "if" direction. ■

After proving the above conditions, we are ready to prove Theorem 2 which states that FPMinimumCapacity returns a valid value for finite k and an exact value for $k = \infty$.

Proof of Theorem 2 We will show that $\Theta^{\min}$ returned from FPMinimumCapacity corresponds to the value on the right-hand side of Equation (37) of Lemma 13. The loop from Line 4 to 15 iterates through each consecutive pair of values t_a and t_{a+1} in $\widehat{S}_i(\tau, k)$ to find optimal capacity for each line segment defined by the endpoints $(t_a, \bar{D}_{t_a})$ and $(t_{a+1}, D_{t_{a+1}})$. It sets variables corresponding to $\widehat{W}_i(t_a)$ and $\widehat{W}_i(t_{a+1})$ in Lines 5 and 6 respectively. Then, in the next few lines it sets four different values to ℓ (based on ℓ_1 and ℓ_2, defined in Equations (14) and (15)) and evaluates $\Phi_j(\cdot)$ according to Lemma 12 to compute $\Theta^*_\ell(\cdot)$ for each of the four integer values of ℓ. Therefore, $\Theta^{\min}_{t_a}$, set in Line 13, equals $\Theta^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)$ by Lemma 5. At the end of this loop it sets $\Theta^{\min}_i$ to be the minimum of $\Theta^{\min}_{t_a}$ and $\Theta^{\min}_i$ (Line 14). Thus, once the inner loop is executed for all $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$, $\Theta^{\min}_i$ contains the minimum of all $\Theta^{\min}_{t_a}$ values. The outer loop from Line 2 to Line 17 finds $\Theta^{\min}_i$ for all task τ_i in τ. Finally, in Line 16, $\Theta^{\min}$ is set to the maximum of $U(\tau) \cdot \Pi$ and $\Theta^{\min}_i$ over all values τ_i in τ.

By Lemma 13, $\widehat{W}_i(t) \leq \mathsf{sbf}((\Pi, \Theta^{\min}, \Delta), t)$ for some $t \in (0, d_i]$ and $U(\tau) \leq \frac{\Theta}{\Pi}$, by Lemma 1, $W_i(t) \leq \widehat{W}_i(t)$. These implies $W_i(t) \leq \mathsf{sbf}((\Pi, \Theta^{\min}, \Delta), t)$ for all $t \geq 0$ which is the schedulability condition given by Theorem 1. Therefore, τ will always meet all deadlines when scheduled by fixed-priority scheduling upon $\Omega = (\Pi, \Theta^{\min}, \Delta)$. When $k = \infty$, $\widehat{W}_i(t)$ equals $W_i(t)$ for all $t \geq 0$; in this case, $\Theta^{\min}$ equals $\Theta^*(\Pi, \Delta, \tau)$ (i.e., $\Theta^{\min}$ is exact capacity). ■

5 An Approximation Scheme

In the previous section, we have shown that FPMinimumCapacity gives a valid answer when k is finite and an exact answer when k is infinite. In this section, we show that as k increases, the guaranteed accuracy of FPMinimumCapacity increases along with its running time. Theorem 3 presents the tradeoff between accuracy and computational complexity, in terms of k.

Theorem 3 *Given Π, Δ, τ, and $k \in \mathbb{N}^+$, the procedure* FPMinimumCapacity *returns $\Theta^{\min}$ such that*

$$\Theta^*(\Pi, \Delta, \tau) \leq \Theta^{\min} \leq \left(\frac{k+1}{k} \right) \cdot \Theta^*(\Pi, \Delta, \tau).$$

Furthermore, FPMinimumCapacity (Π, Δ, τ, k) *has time complexity $O(kn^2 \log n)$*

The following corollary quantifying our FPTAS is immediately obtainable from Theorem 3, by substituting a value for k dependent on the accuracy parameter ϵ ($k = \lceil \frac{1}{\epsilon} \rceil$).

Corollary 5 *Given* Π*,* Δ*,* τ*, and* $\epsilon > 0$*, the procedure* FPMINIMUMCAPACITY $\left(\Pi, \Delta, \tau, \lceil \frac{1}{\epsilon} \rceil\right)$ *returns* $\Theta^{\min}$ *such that*

$$\Theta^*(\Pi, \Delta, \tau) \leq \Theta^{\min} \leq (1+\epsilon) \cdot \Theta^*(\Pi, \Delta, \tau).$$

Furthermore, FPMINIMUMCAPACITY $\left(\Pi, \Delta, \tau, \lceil \frac{1}{\epsilon} \rceil\right)$ *has time complexity* $O\left(\frac{n^2 \log n}{\epsilon}\right)$.

To prove Theorem 3, we need to prove two additional lemmas.

Lemma 14 *Given* Π*,* Δ*, and pair of consecutive pair of values* $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$*, the following is true for all* $k, \ell (\in \mathbb{N}^+)$*, and* $\alpha (\in [0,1])$*,*

$$\Theta_\ell^*\left(\Pi, \Delta, \left\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \right\rangle\right) \leq \left(\tfrac{k+1}{k}\right) \cdot \Theta_\ell^*\left(\Pi, \Delta, \left\langle \left(t_a, \tfrac{k \cdot \bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \tfrac{k \cdot D_{t_{a+1}}}{k+1}\right), \tfrac{k \cdot \alpha}{k+1} \right\rangle\right). \tag{38}$$

Proof: By Lemma 12, $\Theta_\ell^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)$ must be equal to one of Φ_1, Φ_2 or Φ_3 according to the value of ℓ. We will show that for each of the three possibilities, Equation (38) must hold.

If $\Theta_\ell^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)$ is equal to $\Phi_1(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell, \Pi, \Delta)$ (i.e., $\frac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell+1}$), then $\ell \geq \lfloor \ell_1 \rfloor + 1$ by Lemma 12. This implies by definition of ℓ_1,

$$\begin{aligned}
\ell &\geq \left\lfloor \tfrac{(t_{a+1}-\Delta) + \sqrt{(t_{a+1}-\Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi} \right\rfloor + 1 \\
&> \tfrac{(t_{a+1}-\Delta) + \sqrt{(t_{a+1}-\Delta)^2 + 4\Pi D_{t_{a+1}}}}{2\Pi} \\
&> \tfrac{2(t_{a+1}-\Delta)}{2\Pi} \\
&= \tfrac{t_{a+1}-\Delta}{\Pi}.
\end{aligned}$$

Thus, $\ell\Pi + \Delta - t_{a+1} \geq 0$. By Lemma 12 and $\ell \geq \lfloor \ell_1 \rfloor + 1$,

$$\begin{aligned}
&\Theta_\ell^*\left(\Pi, \Delta, \left\langle \left(t_a, \frac{k \cdot \bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \frac{k \cdot D_{t_{a+1}}}{k+1}\right), \frac{k \cdot \alpha}{k+1} \right\rangle\right) \\
&\quad = \Phi_1\left(\left\langle \left(t_a, \tfrac{k \cdot \bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \tfrac{k \cdot D_{t_{a+1}}}{k+1}\right), \tfrac{k \cdot \alpha}{k+1} \right\rangle, \ell, \Pi, \Delta\right) \\
&\quad = \tfrac{\frac{k \cdot D_{t_{a+1}}}{k+1} - t_{a+1} + \ell\Pi + \Delta}{\ell+1} \\
&\quad \geq \tfrac{\frac{k}{k+1} \cdot D_{t_{a+1}} + \frac{k}{k+1} \cdot (\ell\Pi + \Delta - t_{a+1})}{\ell+1} \\
&\quad \geq \left(\tfrac{k}{k+1}\right) \cdot \left(\tfrac{D_{t_{a+1}} - t_{a+1} + \ell\Pi + \Delta}{\ell+1}\right) \\
&\quad = \left(\tfrac{k}{k+1}\right) \cdot \Theta_\ell^*\left(\Pi, \Delta, \left\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \right\rangle\right).
\end{aligned}$$

In this case, Equation (38) holds.

If $\Theta_\ell^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle)$ is equal to $\Phi_2(\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle, \ell, \Pi, \Delta)$ (i.e, $\frac{\bar{D}_{t_a}}{\ell}$), then $\ell \leq \lceil \ell_2 \rceil - 1$ by Lemma 12. Lemma 12 also implies

$$\begin{aligned}
&\Theta_\ell^*\left(\Pi, \Delta, \left\langle \left(t_a, \frac{k \cdot \bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \frac{k \cdot D_{t_{a+1}}}{k+1}\right), \frac{k \cdot \alpha}{k+1} \right\rangle\right) \\
&\quad = \Phi_2\left(\left\langle \left(t_a, \tfrac{k \cdot \bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \tfrac{k \cdot D_{t_{a+1}}}{k+1}\right), \tfrac{k \cdot \alpha}{k+1} \right\rangle, \ell, \Pi, \Delta\right) \\
&\quad \geq \tfrac{\frac{k \cdot \bar{D}_{t_a}}{k+1}}{\ell} \\
&\quad = \left(\tfrac{k}{k+1}\right) \cdot \Theta_\ell^*\left(\Pi, \Delta, \left\langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \right\rangle\right).
\end{aligned}$$

Finally, if $\Theta^*_\ell(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle)$ is equal to $\Phi_3(\langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle, \ell, \Pi, \Delta)$ (i.e., $\frac{\bar{D}_{t_a}+\alpha(\ell\Pi+\Delta-t_a)}{\ell+\alpha}$),then $\lceil \ell_2 \rceil \leq \ell \leq \lfloor \ell_1 \rfloor$ by Lemma 12. Lemma 12 also implies that

$$\begin{aligned}
&\Theta^*_\ell\left(\Pi, \Delta, \left\langle\left(t_a, \frac{k\cdot\bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \frac{k\cdot D_{t_{a+1}}}{k+1}\right), \frac{k\cdot\alpha}{k+1}\right\rangle\right) \\
&= \Phi_3\left(\left\langle\left(t_a, \tfrac{k\cdot\bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \tfrac{k\cdot D_{t_{a+1}}}{k+1}\right), \tfrac{k\cdot\alpha}{k+1}\right\rangle, \ell, \Pi, \Delta\right) \\
&\geq \frac{\frac{k\cdot\bar{D}_{t_a}}{k+1} - \left(\frac{k\cdot\alpha}{k+1}\right)(\ell\Pi+\Delta-t_a)}{\ell+\left(\frac{k\cdot\alpha}{k+1}\right)} \\
&\geq \left(\tfrac{k}{k+1}\right)\cdot\left(\frac{\bar{D}_{t_a}-\alpha(\ell\Pi+\Delta-t_a)}{\ell+\alpha}\right) \\
&= \left(\tfrac{k}{k+1}\right)\cdot\Theta^*_\ell\left(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle\right).
\end{aligned}$$

■

Lemma 15 *Given* Π, Δ, $\tau_i \in \tau$, *and* $k \in \mathbb{N}^+$, *there exists consecutive pair of values* $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$ *such that,*

$$\Theta^*(\Pi, \Delta, \tau) \geq \Theta^*\left(\Pi, \Delta, \left\langle\left(t_a, \frac{k\cdot\bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \frac{k\cdot D_{t_{a+1}}}{k+1}\right), \frac{k\cdot\alpha}{k+1}\right\rangle\right). \tag{39}$$

Proof:

Let Θ_{RHS} denote the right-hand side of Equation (39). By definition of $\Theta^*(\Pi, \Delta, \tau)$ and Theorem 1, for all $\tau_i \in \tau$, there exist $t \in (0, d_i]$ such that

$$W_i(t) \leq \mathsf{sbf}((\Pi, \Theta^*(\Pi, \Delta, \tau), \Delta), t). \tag{40}$$

Now consider any pair of consecutive values $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$. By Lemma 1, we have, for all $t \in (t_a, t_{a+1}]$,

$$\begin{aligned}
&\left(\frac{k+1}{k}\right)\cdot W_i(t) \\
&= \left(\tfrac{k+1}{k}\right)\cdot\left(e_i + \textstyle\sum_{j=1}^{i-1}\mathsf{RBF}(\tau_j, t)\right) \\
&\geq e_i + \left(\tfrac{k+1}{k}\right)\cdot\textstyle\sum_{j=1}^{i-1}\mathsf{RBF}(\tau_j, t) \\
&\geq e_i + \left(\tfrac{k+1}{k}\right)\cdot\left(\textstyle\sum_{j=1}^{i-1}\delta(\tau_i, t)\cdot\tfrac{k}{k+1}\right) \\
&= \widehat{W}_i(t)
\end{aligned} \tag{41}$$

Combining the inequalities of Equations (40) and (41) gives us, for all $t \in (t_a, t_{a+1}]$,

$$\begin{aligned}
&\mathsf{sbf}((\Pi, \Theta^*(\Pi, \Delta, \tau), \Delta), t) \\
&\geq \tfrac{k}{k+1}\cdot\widehat{W}_i(t).
\end{aligned} \tag{42}$$

Lemma 4 and Equation (42) imply that there exists $\ell \in \mathbb{N}$ and $(t, D_t) \in \left\langle\left(t_a, \frac{k\cdot\bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \frac{k\cdot D_{t_{a+1}}}{k+1}\right), \frac{k\cdot\alpha}{k+1}\right\rangle$ such that

$$(t, D_t) \in \mathcal{F}_\ell(\Pi, \Theta^*(\Pi, \Delta, \tau), \Delta).$$

The above expression and Definition 5 implies

$$\Theta^*_\ell\left(\Pi, \Delta, \left\langle\left(t_a, \frac{k\cdot\bar{D}_{t_a}}{k+1}\right), \left(t_{a+1}, \frac{k\cdot D_{t_{a+1}}}{k+1}\right), \frac{k\cdot\alpha}{k+1}\right\rangle\right) \leq \Theta^*(\Pi, \Delta, \tau).$$

The lemma follows from the expression above and Lemma 5.

■

We find the following corollary by combining Lemmas 14, 15 and 5.

Corollary 6 *Given* Π, Δ, $k \in \mathbb{N}^+$, *and* τ_i, *there exists consecutive pair of values* $t_a, t_{a+1} \in \widehat{S}_i(\tau, k)$,

$$\begin{aligned}
&\left(\frac{k+1}{k}\right)\cdot\Theta^*(\Pi, \Delta, \tau) \\
&\geq \inf_{\ell\in\mathbb{N}^+}\left\{\Theta^*_\ell\left(\Pi, \Delta, \langle(t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha\rangle\right)\right\}.
\end{aligned} \tag{43}$$

Now, we are ready to give the proof of Theorem 3.

Proof of Theorem 3 We already proved the first part in Theorem 2; now we must prove the second part of the inequality. From our algorithm, the value of $\Theta^{\min}$ can be either equal to $\Pi \cdot U(\tau)$ or greater than this term. If $\Theta^{\min} = \Pi \cdot U(\tau)$, Theorem 1 implies that $\Theta^*(\Pi, \Delta, \tau)$ must be at least $U(\tau) \cdot \Pi$. For this case, the second inequality follows, since $\frac{k+1}{k} \geq 1$ for all $k \in \mathbb{N}^+$. Now consider the case when $\Theta^{\min} > \Pi \cdot U(\tau)$.

$$\Theta^{\min} = \max_{\tau_i \in \tau} \left\{ \min_{t_a, t_{a+1} \in \widehat{S}_i} \left\{ \Theta^*(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle) \right\} \right\}$$

according to Theorem 2 and Lemma 13. By Lemma 5, this is equivalent to

$$\Theta^{\min} = \max_{\tau_i \in \tau} \left\{ \min_{t_a, t_{a+1} \in \widehat{S}_i} \left\{ \inf_{\ell \in \mathbb{N}^+} \left\{ \Theta^*_\ell \left(\Pi, \Delta, \langle (t_a, \bar{D}_{t_a}), (t_{a+1}, D_{t_{a+1}}), \alpha \rangle \right) \right\} \right\} \right\}.$$

Applying Corollary 6, we find,

$$\Theta^{\min} \leq \max_{\tau_i \in \tau} \left\{ \left(\frac{k+1}{k} \right) \cdot \Theta^*(\Pi, \Delta, \tau) \right\}.$$

From this and the definition of $\Theta^*(\Pi, \Delta, \tau)$ the second inequality of this theorem follows.

■

6 Simulations

This section shows the simulation results for our proposed algorithm, and compares it with the exact algorithm [11] and the sufficient algorithm from [25], which only uses the task system utilization and relative period ratios to determine the capacity. The simulation parameters and value ranges are similar to [15] and shown below:

1. The number of tasks in a task system τ is $2, 4, 8, 16, 32$ or 64.
2. The system utilization $U(\tau)$ is taken from the range $[0.1, 0.8]$ at 0.05-increments and individual task utilizations u_i are generated using UUniFast algorithm [7].
3. Each sporadic task $\tau = (e_i, d_i, p_i)$ has a period p_i uniformly drawn from the interval $[5, 1000]$. The execution time e_i is set to $u_i.p_i$. For each task, we assume d_i equals p_i.
4. The component level scheduling algorithm is DM.
5. The value of k is set to $3, 4$, or 5 (equivalent to $\epsilon = \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$). Π is set to $5, 10$, or 15; Δ is equal to Π.

For each simulation, given task system size n and system utilization $U(\tau)$, we randomly generate taskset parameters u_i, p_i, and e_i for each task τ_i. We execute the exact algorithm [11], the sufficient algorithm [25] and FPMINIMUMCAPACITY to generate exact, sufficient and approximate capacity, respectively. Each point in the following plots represents the mean of 1000 simulation results. We show the results for only one combination of parameters.

In Figure 4, the relative error in the calculation of capacity for our algorithms is plotted as a function of task system utilization. Relative error is defined as follows: $\frac{\Theta - \Theta^*}{\Theta^*}$. In this case, the estimated capacity Θ is either the sufficient capacity (denoted by $\widehat{\Theta}$) or approximate capacity (denoted by $\Theta^{\min}$). In the graph, the solid-line curve represents relative error for $\widehat{\Theta}$ and the dotted-line curve represents relative error for $\Theta^{\min}$. For FPMINIMUMCAPACITY, the mean relative error is less than 5%, whereas for the sufficient algorithm it ranges from 30% to 95%. The 95% confidence intervals are shown.
For the sufficient algorithm, relative error is very high due to the fact that the algorithm overestimates capacity.

As we have mentioned, the runtime complexity of FPMINIMUMCAPACITY depends entirely on the size of the testing set. Figure 5 shows a comparison between testing set sizes for the exact algorithm ($\mathsf{TS} = \sum_{i=1}^{n} |S_i|$) and the approximate algorithm ($\widetilde{\mathsf{TS}} = \sum_{i=1}^{n} |\widehat{S}_i|$). The solid-line curve in the graph represents $\widetilde{\mathsf{TS}}$ and the dotted-line curve represents TS. As we know from our algorithm, $\widetilde{\mathsf{TS}}$ only depends on the input k and taskset size n (which is constant for our graph). On the other hand, TS may be pseudo-polynomial since it depends on the periods of the tasks. Therefore, we can conclude that FPMINIMUMCAPACITY reduces the pseudo polynomial time complexity of the exact algorithm to polynomial time while still maintaining a lower relative error.

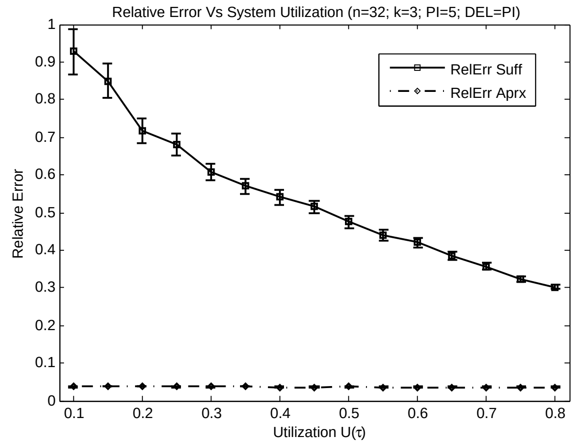

Figure 4. Relative Error vs System Utilization

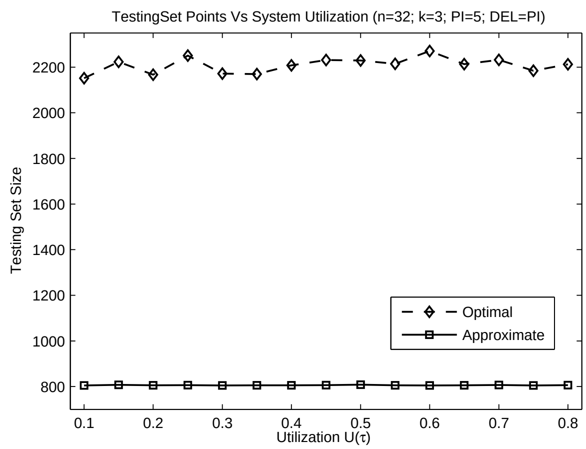

Figure 5. Testing Set size vs System Utilization

7 Conclusions and Future Work

In this paper, we have extended the results of [15] for compositional real-time systems with fixed priority component level scheduling algorithms. We devised a fully polynomial time approximation scheme (FPTAS) for the minimization of interface bandwidth (MIB-RT) problem of explicit-deadline periodic (EDP) resource model in this case. In this model, given fixed period and deadline of the EDP resource, for any sporadic task system our algorithm returns bandwidth that is at most a factor of $(1+\epsilon)$ greater than the optimal minimum bandwidth, for any $\epsilon > 0$. We showed that our algorithm has a polynomial time complexity in terms of the number of tasks in the task system n and the approximation parameter $1/\epsilon$, whereas exact algorithms for MIB-RT problem on fixed priority periodic resources may require pseudo polynomial time [11, 25] depending on the task parameters (i,e. task deadline or period) of the task system. We verified our result by running simulation over synthetically generated tasks, and showed that our approximation algorithm improves performance over the sufficient algorithm [25] by effectively reducing relative error. Also the algorithm closely approximates the bandwidth from the exact algorithm regardless of the task parameters while maintaining polynomial time complexity.

Our result in this paper is for fixed priority scheduling algorithms of constrained deadline sporadic task systems. Future direction from this paper may be to extend this work to more general task models such as fixed priority task system with arbitrary deadlines, hybrid priority task systems etc. Furthermore, the approximation algorithms for MIB-RT on uniprocessor frameworks may be applicable to multiprocessor compositional frameworks (e.g., [22]) as well as to other compositional resource models.

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