Approximate Bandwidth Allocation for Compositional Real-Time Systems (WSU-CS Technical Report Version)

Nathan Fisher* Farhana Dewan

Abstract

Allocation of bandwidth among components is a fundamental problem in compositional real-time systems. State-of-the-art algorithms for bandwidth allocation use either exponential-time or pseudo-polynomial-time techniques for exact allocation, or linear-time, utilization-based techniques which may over-provision bandwidth. In this paper, we develop a fully-polynomial-time approximation scheme (FPTAS) for allocating bandwidth for sporadic task systems scheduled by earliest-deadline first (EDF) upon an Explicit-Deadline Periodic (EDP) resource. Our algorithm takes, as parameters, the task system and an accuracy parameter $\varepsilon > 0$*, and returns a bandwidth which is guaranteed to be at most a factor* $(1+\varepsilon)$ *more than the optimal minimum bandwidth required to successfully schedule the task system. Furthermore, the algorithm has time complexity that is polynomial in the number of tasks and* $1/\varepsilon$*. Via simulations over randomly-generated task systems, we have observed a several orders of magnitude decrease in runtime and a small relative error when comparing our proposed algorithm with the exact algorithm, even for medium-sized values of* ε *(e.g.,* $\varepsilon \approx .3$*).*

1 Introduction

Component-based design is widely practiced in system design and development due to the remarkable benefits obtained from decomposing a complex system into simpler components. *Component abstraction*, a central goal of component-based design, permits each component to hide internal complexity and details from developers of other components and only exposes information necessary to use the component via an interface. Recently, component-based design for real-time systems has received considerable attention, as numerous frameworks for compositional real-time systems have been proposed (e.g., [9, 11, 21, 1]).

*N. Fisher and F. Dewan are with Department of Computer Science, Wayne State University in Detroit, MI, USA. Corresponding Author's Email:fishern@cs.wayne.edu.

Most real-time compositional frameworks provide a *real-time interface* through which a component may express its temporal requirements (e.g., processing time requirements and deadlines). An important attribute of a real-time interface is the *interface bandwidth*. The interface bandwidth simultaneously quantifies the fraction of the total system resource supply that a component C will require to meet its real-time constraints and the component C's "interference" on the resource supply provided to other system components. Thus, an important fundamental problem in design and analysis of compositional real-time systems is the *minimization of real-time interface bandwidth* (MIB-RT). In this paper, we address the MIB-RT problem for a real-time compositional framework known as the *explicit-deadline periodic resource model* (EDP) [10].

In the EDP model, a resource Ω is characterized by a three-tuple (Π, Θ, Δ). The interpretation of such a resource is that a component C executed upon Ω is guaranteed Θ units of processing resource supply for successive Π-length intervals (given some initial starting time). Furthermore, the Θ units of resource supply must be provided within Δ $(\leq \Pi)$ time units after the start of the Π-length interval. For $\Omega = (\Pi, \Theta, \Delta)$, Π is referred to as the *period of repetition*, Θ is the *capacity*, and Δ is the *relative deadline*. The MIB-RT problem for the EDP model (under EDF-scheduling) can be stated as follows: *for any component* C *and fixed values of* Π *and* Δ*, determine a capacity* $\widehat{\Theta}$ *such that* $\frac{\widehat{\Theta}}{\Pi}$ *is minimized and* C *is* EDF*-schedulable upon a resource* $\Omega = (\Pi, \widehat{\Theta}, \Delta)$. (Note that Shin and Lee call this value the *minimum periodic capacity* [19]). The interface bandwidth[1] of C (given fixed Π and Δ) is $\frac{\widehat{\Theta}}{\Pi}$.

The MIB-RT problem has previously been studied for the EDP model where each component is represented by a *sporadic task system* [16]. Easwaran et al. [10] obtain exact solutions to MIB-RT; i.e., if the bandwidth provided by the system to component C is less than the exact solution to MIB-RT (i.e., minimum bandwidth), then some real-time constraint will be violated for C. However, this solution is

[1] We abuse terminology slightly and allow (Π, Θ, Δ) describe both the resource Ω and an interface for C.

based upon exact schedulability techniques for uniprocessor real-time systems [5, 13], which may be computationally expensive. Shin and Lee [21] have obtained $O(n)$-time[2], sufficient solutions to MIB-RT when Π equals Δ. The advantage of this approach is that bandwidth allocation may be determined quickly for a component C. Efficient determination of bandwidth is often a critical issue in compositional real-time systems where components may dynamically join the system. However, the disadvantage is that the allocated bandwidth may be more than necessary – leading to a potential waste of processing resources. Specifically, our previous analysis [12] of Shin and Lee's linear-time algorithm showed that there exists sporadic task systems that would cause the algorithm to return a bandwidth that is a factor of 1.5 greater than the optimal bandwidth. (However, it is also shown that the most that the returned bandwidth could exceed optimal is by a factor of 3).

§**Our Contribution.** Our objective is to address the current gap between computationally-expensive, exact solutions and computationally-inexpensive, inexact solutions for MIB-RT problem in the context of the EDP resource model and components composed of sporadic tasks. Under this setting, we develop a parametric approximation algorithm. Our parametric algorithm allows the component designer to pre-specify an arbitrary level of accuracy in obtaining a solution to MIB-RT; however, the desired level of accuracy has a quantifiable trade-off with the efficiency of obtaining the solution to MIB-RT. Specifically, our algorithm gives the following guarantee.

> Given Π, Δ, task system τ, and accuracy parameter $\varepsilon > 0$. Let $\Theta^*(\Pi,\Delta,\tau)$ be the optimal minimum capacity for τ to be EDF-schedulable upon $\Omega^* = (\Pi, \Theta^*(\Pi,\Delta,\tau), \Delta)$. If our algorithm returns $\widehat{\Theta}$ for the given parameters, then $\Theta^*(\Pi,\Delta,\tau) \leq \widehat{\Theta} \leq (1+\varepsilon)\cdot\Theta^*(\Pi,\Delta,\tau)$. Furthermore, our algorithm runs in time polynomial in the number of tasks in τ and $\frac{1}{\varepsilon}$.

In other words, our algorithm is a fully-polynomial-time approximation schemes (FPTAS) [22] for the MIB-RT problem. The $(1+\varepsilon)$ factor is called the *approximation ratio* of the produced solution. We also show, via simulation, that our algorithm is quite accurate over synthetically generated task systems even for medium-sized values of ε.

§**Organization.** The remainder of the paper is organized as follows. We briefly review prior related research on MIB-RT for various compositional real-time frameworks in Section 2. We give necessary background and notation, in Section 3, to describe the MIB-RT problem in the context of a sporadic task system executing upon explicit-deadline periodic resource. We give our algorithm and prove its correctness in Section 4. Approximation ratio results for our proposed algorithm are contained in Section 5; furthermore, we show that our algorithm is an FPTAS. Simulations results comparing our algorithm with both previously-known exact and sufficient algorithms are given in Section 6.

[2] Shin and Lee's algorithm is $O(1)$, if the task system utilization and smallest task period values are precomputed.

2 Related Work

Over the past decade, many different real-time compositional models have been proposed, since the original seminal works in *real-time open environments* by Deng and Liu [9] and *resource kernels* by Rajkumar et al. [17]. The MIB-RT problem has been a well-studied problem in each of the proposed compositional models. In this section, we give a very high-level (and abridged) survey of some of the prior work on MIB-RT for compositional real-time systems. Feng and Mok [11] proposed the concept of *temporal partitions* to support hierarchical sharing of a processing resource. Shin et al. [21, 19] proposed the related *periodic resource model* to characterize the supply guaranteed to any component in compositional system (along with algorithms for MIB-RT described in the introduction). For components scheduled by fixed-priority on temporal partitions, Lipari and Bini [14], developed an exact, pseudo-polynomial-time algorithm for MIB-RT. While Almeida and Pedreiras [3] developed sufficient, polynomial-time bandwidth allocation techniques for fixed-priority scheduling upon temporal partitions.

Recently, researchers have also focused on characterizing components by processor-demand curves which describe the minimum amount of processing that a component requires over any interval. For example, Wandeler and Thiele [23] proposed the concept of *interface-based design* which uses real-time calculus [8] to compute demand curves and service curves for each component in a compositional real-time system. In another demand-based model, Albers et al. [1] have developed parametric algorithms for MIB-RT (without known approximation ratios) for the *hierarchical event stream model*. Thus, for a variety of both partition-based models and demand-supply curve models, relatively efficient, sufficient algorithms for MIB-RT have been proposed; however, we are currently unaware of any work on obtaining polynomial-time algorithms with constant-factor approximation ratios (other than our preliminary work on obtaining such ratios for the periodic resource model [12]). The goal of our work is to fill this needed gap by obtaining an FPTAS for MIB-RT in the periodic resource model.

Expression	Description
τ	Sporadic Task System
τ_i	Sporadic Task
n	Number of Tasks in τ
e_i	Execution Requirement of τ_i
d_i	Relative Deadline of τ_i
p_i	Period of τ_i
u_i	Utilization of τ_i
$U(\tau)$	System Utilization
$\mathsf{dbf}(\tau_i,t)$	Demand-Bound Function for τ_i
$\mathsf{DBF}(\tau,t)$	Demand-Bound Function for τ
$\widetilde{\mathsf{dbf}}(\tau_i,t,k)$	Approx. Demand-Bound Function for τ_i
$\widetilde{\mathsf{DBF}}(\tau,t,k)$	Approx. Demand-Bound Function for τ
k	Number of Steps for $\widetilde{\mathsf{dbf}}$
$\mathsf{TS}(\tau)$	Exact Testing Set for τ
$\widetilde{\mathsf{TS}}(\tau,k)$	Approx. Testing Set for τ
$P(\tau)$	Upper bound on Testing Set
$\langle(t,D_t),\alpha\rangle$	Half Line Originating at (t,D_t) with slope α
$\psi(\tau,t,k)$	Slope of $\widetilde{\mathsf{DBF}}(\tau,t,k)$ at t
Ω	EDP Resource
Π	Period of Repetition for Ω
Θ	Capacity of Ω
Δ	Deadline of Ω
$\mathsf{sbf}(\Omega,t)$	Supply-Bound Function for Ω
$\mathsf{lsbf}(\Omega,t)$	Lower Supply-Bound Function for Ω
$\Theta^*(\Pi,\Delta,\tau)$	Minimum Capacity for Ω given τ
$\Theta^*(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$	Minimum Capacity for Half-Line
$\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$	ℓ-Minimum Capacity for Half-Line
$\mathscr{F}_\ell(\Pi,\Theta,\Delta)$	ℓ-Feasibility Region for Ω

Table 1. Major notation used in this paper.

3 Models and Notation

In this section, we present background and notation for the task model, workload functions, and periodic resource model that we use throughout the paper. Given the large amount of notation needed, we have collected all major notation that is used repeatedly in Table 1.

§**Sporadic Task Model.** A **sporadic task** $\tau_i = (e_i, d_i, p_i)$ is characterized by a *worst-case execution requirement* e_i, a *(relative) deadline* d_i, and a *minimum inter-arrival separation* p_i, which is, for historical reasons, also referred to as the *period* of the task. Such a sporadic task generates a potentially infinite sequence of jobs, with successive job-arrivals separated by at least p_i time units. Each job has a worst-case execution requirement equal to e_i and a deadline that occurs d_i time units after its arrival time. A *sporadic task system* $\tau \stackrel{\text{def}}{=} \{\tau_1, \ldots, \tau_n\}$ is a collection of n such sporadic tasks. A useful metric for a sporadic task τ_i is the *task utilization* $u_i \stackrel{\text{def}}{=} e_i/p_i$. The system utilization is denoted $U(\tau) \stackrel{\text{def}}{=} \sum_{\tau_i\in\tau} u_i$.

§**Workload Functions.** For determining schedulability of a sporadic task system, it is often useful to quantify the maximum amount of execution that must complete over any given interval. For this purpose, researchers [6] have derived the *demand-bound function*, defined below.

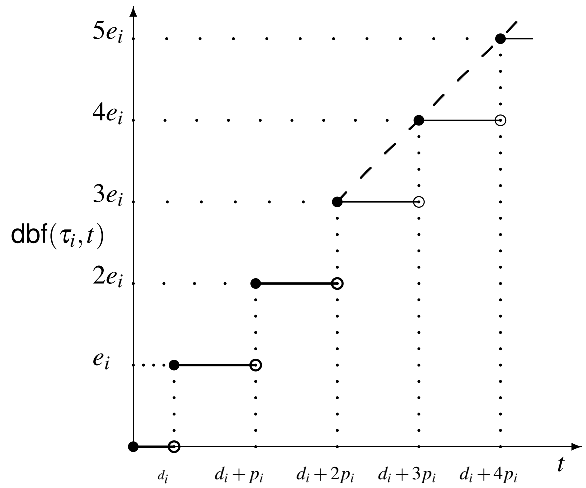

Figure 1. The step function denotes a plot of $\mathsf{dbf}(\tau_i,t)$ as a function of t. The dashed line represents the function $\widetilde{\mathsf{dbf}}(\tau_i,t,k)$, approximating $\mathsf{dbf}(\tau_i,t)$. $\widetilde{\mathsf{dbf}}(\tau_i,t,k)$ is equal to $\mathsf{dbf}(\tau_i,t)$ for all $t < d_i + (k-1)p_i$ (k equals three in the above graph).

Definition 1 (Demand-Bound Function) *For any $t > 0$ and task τ_i, the* **demand-bound function** *(dbf) quantifies the maximum cumulative execution requirement of all jobs of τ_i that could have both an arrival time and deadline in any interval of length t. Baruah et al. [6] have shown that, for sporadic tasks, dbf can be calculated as follows.*

$$\mathsf{dbf}(\tau_i,t) = \max\left(0, \left\lfloor \frac{t-d_i}{p_i} \right\rfloor + 1\right)\cdot e_i. \tag{1}$$

Figure 1 gives a visual depiction of the demand-bound function for a sporadic task τ_i. Observe from the above definition and Figure 1 that the dbf is a right continuous function with discontinuities at time points of the form $t \equiv d_i + a\cdot p_i$ where $a \in \mathbb{N}$. Let $\mathsf{DBF}(\tau,t) \stackrel{\text{def}}{=} \sum_{\tau_i\in\tau} \mathsf{dbf}(\tau_i,t)$. It has been shown [6] that condition $\mathsf{DBF}(\tau,t) \le t, \forall t \ge 0$ is necessary and sufficient for sporadic task system τ to be EDF-schedulable upon a preemptive uniprocessor platform of unit speed. Furthermore, it has also been shown that the aforementioned condition needs to be verified at only time points in the following *ordered* set (with elements are in non-decreasing order):

$$\mathsf{TS}(\tau) \stackrel{\text{def}}{=} \bigcup_{\tau_i\in\tau} \{t \equiv d_i + a\cdot p_i \mid (a\in\mathbb{N}) \wedge (t \le P(\tau))\}. \tag{2}$$

where $P(\tau)$ is an upper bound on the maximum time instant that the schedulability condition must be verified at. For

EDF-scheduled sporadic task systems on preemptive unit-speed processors, $P(\tau)$ is at most $\mathsf{lcm}_{\tau_i \in \tau}\{p_i\}$. The above set is known as the **testing set** for sporadic task system τ. For any $t_a \in \mathsf{TS}(\tau)$, $t_a \le t_{a+1}$; if t_a is the last element of the set, we use the convention that t_{a+1} equals ∞. Also, we will assume that t_0 is equal to zero.

Albers and Slomka [2] proposed the following approximation to dbf to reduce the number of discontinuities (and, thus, points in the testing set).

$$\widetilde{\mathsf{dbf}}(\tau_i, t, k) \stackrel{\text{def}}{=} \begin{cases} \mathsf{dbf}(\tau_i, t), & \text{if } t < d_i + (k-1)p_i; \\ u_i \cdot (t - d_i) + e_i, & \text{otherwise.} \end{cases} \tag{3}$$

The main intuition behind $\widetilde{\mathsf{dbf}}(\tau_i, t, k)$ is that it "tracks" dbf for exactly k discontinuities (i.e., "steps"). After k discontinuities, $\widetilde{\mathsf{dbf}}(\tau_i, t, k)$ using a linear interpolation of the subsequent discontinuous points (with slope equal to u_i). The steps with the thick lines and the sloped-dotted line in Figure 1 correspond to $\mathsf{dbf}(\tau_i, t, 3)$. We will abuse notation slightly and use the convention that $\mathsf{dbf}(\tau_i, t, \infty)$ corresponds to $\mathsf{dbf}(\tau_i, t)$. Let $\widetilde{\mathsf{DBF}}(\tau, t, k) \stackrel{\text{def}}{=} \sum_{\tau_i \in \tau} \widetilde{\mathsf{dbf}}(\tau_i, t, k)$. Albers and Slomka show [2], for any fixed $k \in \mathbb{N}^+$, the condition $\widetilde{\mathsf{DBF}}(\tau, t, k) \le t, \forall t \ge 0$ is sufficient for sporadic task τ to be EDF-schedulable upon a preemptive uniprocessor platform of unit speed. The ordered testing set of this condition is reduced to

$$\widetilde{\mathsf{TS}}(\tau, k) \stackrel{\text{def}}{=} \bigcup_{\tau_i \in \tau} \{t \equiv d_i + a \cdot p_i \mid (a \in \mathbb{N}) \wedge (a < k) \wedge (t \le P(\tau))\}. \tag{4}$$

In order to obtain a fully polynomial-time approximation scheme for preemptive uniprocessors, Albers and Slomka [2] make the following observation regarding the relationship between dbf and $\widetilde{\mathsf{dbf}}$. We will use this observation for our approximation algorithm presented in Section 5.

Lemma 1 (from [2]) *Given a fixed integer $k \in \mathbb{N}^+$, $\mathsf{dbf}(\tau_i, t) \le \widetilde{\mathsf{dbf}}(\tau_i, t, k) \le \left(\frac{k+1}{k}\right) \mathsf{dbf}(\tau_i, t)$ for all $\tau_i \in \tau$ and $t \in \mathbb{R}_{\ge 0}$.*

In addition to the above observation, we will now derive a technical lemma regarding $\widetilde{\mathsf{DBF}}$. This property will be used in Sections 4 and 5 to obtain an exact and an approximate algorithm for determining the minimum bandwidth of a periodic resource that schedules task system τ. Let $\langle (t, D_t), \alpha \rangle$ denote the half-line in Euclidean space $\mathbb{R}^2$, originating at point $(t, D_t) \in \mathbb{R}^2$ with slope α where $0 \le \alpha \le 1$ (i.e., $\langle (t, D_t), \alpha \rangle = \{(x, y) \in \mathbb{R}^2 \mid (x \ge t) \wedge (y = \alpha(x - t) + D_t))\}$). Additionally, we define the following function $\psi(\tau, t, k)$ which quantifies the slope of the expression $\widetilde{\mathsf{DBF}}(\tau, t, k)$ at any time t. Formally,

$$\psi(\tau, t, k) \stackrel{\text{def}}{=} \sum_{\tau_i \in \tau : t \ge d_i + (k-1)p_i} u_i. \tag{5}$$

Note that $\psi(\tau, t, \infty)$ is zero for all t. The following lemma states that for any element t_a of testing set $\widetilde{\mathsf{TS}}(\tau, k)$, the half-line defined by $\left\langle \left(t_a, \widetilde{\mathsf{DBF}}(\tau, t_a, k)\right), \psi(\tau, t_a, k) \right\rangle$ lower bounds $\widetilde{\mathsf{DBF}}(\tau, t, k)$ for all t at least t_a.

Lemma 2 *Given any fixed $k \in \mathbb{N}^+ \cup \{\infty\}$, $t_a \in \widetilde{\mathsf{TS}}(\tau, k)$, and $t \ge t_a$*

$$\widetilde{\mathsf{DBF}}(\tau, t, k) \ge \psi(\tau, t_a, k) \cdot (t - t_a) + \widetilde{\mathsf{DBF}}(\tau, t_a, k). \tag{6}$$

Furthermore, for $t \in [t_a, t_{a+1})$, Equation 6 satisfies equality.

Proof: Given a k and t_a as defined above, consider $\widetilde{\mathsf{DBF}}(\tau, t, k)$ for any $t \ge t_a$.

$$\begin{aligned}
&\widetilde{\mathsf{DBF}}(\tau, t, k) \\
&\quad = \textstyle\sum_{\tau_j \in \tau} \widetilde{\mathsf{dbf}}(\tau_j, t, k) \\
&\quad = \textstyle\sum_{\tau_j \in \tau : t < d_j + (k-1)p_j} \widetilde{\mathsf{dbf}}(\tau_j, t, k) \\
&\qquad\quad + \textstyle\sum_{\tau_j \in \tau : t \ge d_j + (k-1)p_j} [u_j(t - d_j) + e_j] \\
&\qquad\qquad \text{(by Definition of } \widetilde{\mathsf{dbf}}) \\
&\quad \ge \textstyle\sum_{\tau_j \in \tau : t < d_j + (k-1)p_j} \widetilde{\mathsf{dbf}}(\tau_j, t_a, k) \\
&\qquad\quad + \textstyle\sum_{\tau_j \in \tau : t \ge d_j + (k-1)p_j} [(u_j(t_a - d_j) + e_j) + u_j(t - t_a)] \\
&\qquad\qquad (\widetilde{\mathsf{dbf}} \text{ is monontonically non-decreasing}) \\
&\quad = \textstyle\sum_{\tau_j \in \tau} \widetilde{\mathsf{dbf}}(\tau_j, t_a, k) + \psi(\tau, t_a, k) \cdot (t - t_a) \\
&\qquad\qquad \text{(by Definition of } \widetilde{\mathsf{dbf}} \text{ and } \psi).
\end{aligned}$$

The above series of inequalities show Equation 6. We now show that equality holds for Equation 6 for any time instant between t_a and t_{a+1}. For a task $\tau_j \in \tau$, consider $t, t' \in [d_j + (s-1)p_j, d_j + sp_j)$ where $s \in \mathbb{N}^+$ and $s \le k-1$. For any such t and t', $\widetilde{\mathsf{dbf}}(\tau_j, t, k)$ equals $\mathsf{dbf}(\tau_j, t)$ and $\widetilde{\mathsf{dbf}}(\tau_j, t', k)$ equals $\mathsf{dbf}(\tau_j, t')$, by definition of $\widetilde{\mathsf{dbf}}$ (Equation 3). Furthermore, by Definition 1, $\mathsf{dbf}(\tau_j, t)$ equals $\mathsf{dbf}(\tau_j, t')$ due to the floor in the expression of Equation 1. Thus, for such a t and t', $\widetilde{\mathsf{dbf}}(\tau_j, t, k)$ equals $\widetilde{\mathsf{dbf}}(\tau_j, t', k)$. In the third step of the derivation above the inequality may be replaced by equality for all $t \in [t_a, t_{a+1})$ and τ_j where $t < d_j + (k-1)p_j$, since there exists a $s \in \mathbb{N}^+$ $(s < k-1)$ such that $t, t_a \in [d_j + (s-1)p_j, d_j + sp_j)$. (Otherwise, there would exist a $t_b (= d_\ell + (s-1)p_\ell) \in \widetilde{\mathsf{TS}}(\tau, k)$ for some $\tau_\ell \in \tau$ where $t_a < t_b < t_{a+1}$ which would contradict the ordering of the testing set). This implies that $\widetilde{\mathsf{dbf}}(\tau_j, t, k)$ equals $\widetilde{\mathsf{dbf}}(\tau_j, t_a, k)$ for all such $\tau_j \in \tau$ with $t < d_j + (k-1)p_j$. ■

The next corollary immediately follows by combining Lemmas 1 and 2.

Corollary 1 *Given any fixed $k \in \mathbb{N}^+$, $t_a \in \widetilde{\mathsf{TS}}(\tau,k)$, and $t \geq t_a$,*

$$\mathsf{DBF}(\tau,t) \geq \left(\frac{k}{k+1}\right) \cdot \left[\psi(\tau,t_a,k)\cdot(t-t_a) + \widetilde{\mathsf{DBF}}(\tau,t_a,k)\right]. \tag{7}$$

§**Explicit-Deadline Periodic (EDP) Resource Model.** As mentioned in the introduction the EDP resource model [10] generalizes the periodic resource model proposed by Shin and Lee [19, 20]. An EDP resource, denoted by $\Omega = (\Pi,\Theta,\Delta)$, guarantees that a component C executed upon resource Ω will receive at least Θ units of execution between successive time points in $\{t \equiv t_0 + \ell\Pi \mid \ell \in \mathbb{N}\}$ where t_0 is some initial service start-time t_0 for the periodic resource. Furthermore, the Θ units of service must occur Δ units after each successive time point in the aforementioned set. Obviously, $\Theta \leq \Delta$; for this paper, we will make the simplifying assumption that $\Delta \leq \Pi$, as well. Furthermore, we will assume in this paper that each component C is a sporadic task system[3] τ scheduled by EDF upon Ω. (From now on, we use τ in the context of component C).

Definition 2 (Supply-Bound Function) *For any $t > 0$, the* **supply-bound function** *(sbf) quantifies the minimum execution supply that a component executed upon periodic resource Ω may receive over any interval of length t. Easwaran et al. [10] have quantified the supply bound function for an EDP resource in the following (see Figure 3 for a graphical depiction of the function):*

$$\mathsf{sbf}(\Omega,t) = \begin{cases} y_\Omega\Theta + \max\left(0, t - x_\Omega - y_\Omega\Pi\right), & \text{if } t \geq \Delta - \Theta \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

where $y_\Omega = \left\lfloor \frac{t-(\Delta-\Theta)}{\Pi} \right\rfloor$ and $x_\Omega = (\Pi + \Delta - 2\Theta)$.

EDF-schedulability conditions for EDP resource Ω have been developed [19, 20, 10], as given in the following theorem.

Theorem 1 (from [10]) *A sporadic task system τ is EDF-schedulable upon an EDP resource $\Omega = (\Pi,\Theta,\Delta)$, if and only if,*

$$\left(\mathsf{DBF}(\tau,t) \leq \mathsf{sbf}(\Omega,t),\ \forall t \leq P(\tau)\right) \bigwedge \left(U(\tau) \leq \frac{\Theta}{\Pi}\right) \tag{9}$$

where $P(\tau)$ equals $\mathsf{lcm}_{\tau_i\in\tau}\{p_i\} + \max_{\tau_i\in\tau}\{d_i\}$.

Since sbf is discontinuous at certain points, researchers [20, 10] have defined a linear lower-bound to simplify the supply function:

[3] Observe this is not a restriction on the number of hierarchical levels for our results. Subcomponents may also be represented by sporadic tasks, and our results will apply without change.

MINIMUMCAPACITY(Π,Δ,τ,k)

$\Theta^{\min} \leftarrow U(\tau)\cdot\Pi$
for each $t \in \widetilde{\mathsf{TS}}(\tau,k)$ ▷ (In order)
  ▷ For testing set with $P(\tau)$ defined as in Theorem 1.
    $D_t \leftarrow \widetilde{\mathsf{DBF}}(\tau,t,k)$
    $\alpha \leftarrow \psi(\tau,t,k)$
    $\Theta_t^{\min} \leftarrow \infty$
    for $\ell \leftarrow \max\left\{1, \left\lfloor \frac{t-\Delta}{\Pi} \right\rfloor\right\}$ **to** $\left(\left\lceil \frac{t+\Delta}{\Pi} \right\rceil - 1\right)$
      $\Theta_\ell^{\min} \leftarrow \max \left\{ \begin{array}{r} \alpha\Pi, \\ \frac{D_t - t + \ell\Pi + \Delta}{\ell+1}, \\ \frac{D_t}{\ell}, \\ \frac{D_t + \alpha((\ell+1)\Pi + \Delta - t)}{\ell + 2\alpha} \end{array} \right\}$
      $\Theta_t^{\min} \leftarrow \min\{\Theta_t^{\min}, \Theta_\ell^{\min}\}$
    end (of inner loop)
    $\Theta^{\min} \leftarrow \max\{\Theta^{\min}, \Theta_t^{\min}\}$
end (of outer loop)
return $\Theta^{\min}$

Figure 2. Pseudo-code for determining minimum capacity for a periodic resource given Π, Δ, and τ. Note the algorithm is exact when k equals ∞.

Definition 3 (Lower Supply-Bound Function)

$$\mathsf{lsbf}(\Omega,t) = \frac{\Theta}{\Pi}\left(t - \Pi - \Delta + 2\Theta\right). \tag{10}$$

lsbf is a linear interpolation of the lower portions of the "steps" in the sbf function. It has been shown [20] that $\mathsf{lsbf}(\Omega,t) \leq \mathsf{sbf}(\Omega,t)$ for all $t > 0$. See Figure 3 for a visual depiction of lsbf.

4 An Algorithm for Determining Minimum Capacity

In Figure 2, we give pseudocode for our algorithm, MINIMUMCAPACITY, for determining the minimum capacity required to correctly schedule task system τ according to EDF upon an EDP resource with given Π and Δ parameters. One of the input parameters is $k \in \mathbb{N}^+$. If k is some fixed value, then MINIMUMCAPACITY returns an approximate value for the minimum capacity (as explained in Section 5); however, if the input value of k is equal to ∞, then the returned value will be the actual minimum capacity.

The intuition behind algorithm MINIMUMCAPACITY is as follows: for each value t in the testing set $\widetilde{\mathsf{TS}}(\tau,k)$ find the minimum capacity $\Theta_t^{\min}$ required to guarantee that the half-line $\langle(t, \widetilde{\mathsf{DBF}}(\tau,t,k)), \psi(\tau,t,k)\rangle$ is completely beneath

$\mathsf{sbf}((\Pi, \Theta_t^{\min}, \Delta), t)$. By Lemma 2, this half-line is equal to $\widetilde{\mathsf{DBF}}(\tau, t, k)$ until the next testing set time-point. So, any capacity greater than this minimum capacity $\Theta_t^{\min}$ will ensure that $\widetilde{\mathsf{DBF}}$ falls below sbf up until the next point in the testing set. If we set $\Theta^{\min}$ to be the maximum of all these $\Theta_t^{\min}$ (Line 10 of MINIMUMCAPACITY) and $U(\tau) \cdot \Pi$, then we ensure that each "step" of $\widetilde{\mathsf{DBF}}(\tau, t, k)$ is strictly less than $\mathsf{sbf}((\Pi, \Theta^{\min}, \Delta), t)$ and $U(\tau) \leq \frac{\Theta^{\min}}{\Pi}$. Since $\widetilde{\mathsf{DBF}}(\tau, t, k) \geq \mathsf{DBF}(\tau, t)$ for all t, this implies Theorem 1; thus, τ is EDF-schedulable upon EDP resource $\Omega = (\Pi, \Theta^{\min}, \Delta)$. (Note that if the algorithm returns $\Theta^{\min} = \infty$, we cannot guarantee that τ is schedulable upon any EDP resource with parameters Π and Δ executing upon a unit-speed processor).

§**Algorithm Complexity.** The complexity of MINIMUMCAPACITY depends almost entirely upon the cardinality of $\widetilde{\mathsf{TS}}(\tau, k)$. To see this, observe that the inner loop (Lines 6 to 9) has a constant number of iterations due to the fact that $0 \leq \left(\left\lceil \frac{t+\Delta}{\Pi} \right\rceil - 1\right) - \left\lfloor \frac{t-\Delta}{\Pi} \right\rfloor \leq 2$ (since $0 \leq \Delta \leq \Pi$). The work inside the inner loop takes constant time as well. For the outer loop (Lines 2 to 11), the algorithm iterates (in non-decreasing order) through the testing set $\widetilde{\mathsf{TS}}(\tau, k)$. Using a "heap-of-heaps" described by Mok [15], the time complexity to obtain an element of the testing set is $O(\lg n)$. Since for each element of the testing set, $t \equiv d_i + a p_i$ for some $\tau_i \in \tau$ and $a \in \mathbb{N}$, setting D_t and α (Lines 3 and 4) may be done in constant time on each iteration of the outer loop. (D_t is increased by e_i from prior iteration and α is only increased by u_i, if $a = (k-1)$). Therefore, the runtime complexity of MINIMUMCAPACITY is $O(|\widetilde{\mathsf{TS}}(\tau, k)| \cdot \lg n)$. If $k = \infty$, then $|\widetilde{\mathsf{TS}}(\tau, \infty)| \leq P(\tau)$ which is potentially exponential in the number of tasks. The complexity for exactly determining the minimum capacity is, thus, the same complexity as the test of Theorem 1 on a fixed Ω. Otherwise, if k is a fixed integer, $|\widetilde{\mathsf{TS}}(\tau, k)| \leq kn$ and the complexity is $O(kn \lg n)$.

§**Algorithm Correctness.** To prove the correctness of MINIMUMCAPACITY, we will show the following theorem which states that the value returned by the algorithm (i.e., $\Theta^{\min}$) is at least the optimal minimum capacity value $\Theta^*(\Pi, \Delta, \tau)$. Furthermore, if the input k equals ∞, then the returned capacity is optimal.

Theorem 2 *For all $k \in \mathbb{N}^+ \cup \{\infty\}$,* MINIMUMCAPACITY *returns $\Theta^{\min} \geq \Theta^*(\Pi, \Delta, \tau)$. Furthermore, if $k = \infty$, $\Theta^{\min} = \Theta^*(\Pi, \Delta, \tau)$.*

In order to prove the above theorem, we require some additional definitions. The next definition quantifies the minimum capacity $\Theta(\leq \Delta)$ that is required for sbf to upper-bound a half-line $\langle (t, D_t), \alpha \rangle$. We will use the convention that min returns ∞ on an empty set.

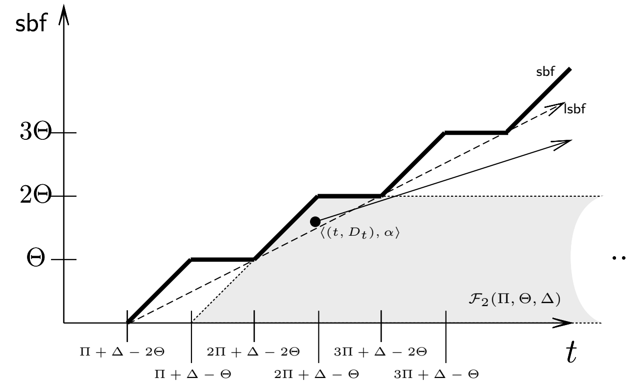

Figure 3. The solid line "step" function is sbf for Ω. The dashed line is the lower supply-bound function lsbf. The shaded region represents the ℓ-Feasibility Region for $\ell = 2$ with element $\langle (t, D_t), \alpha \rangle$.

Definition 4 (Minimum Capacity for $\langle (t, D_t), \alpha \rangle$)

$$\Theta^*(\Pi, \Delta, \langle (t, D_t), \alpha \rangle) \stackrel{def}{=} \min \left\{ \Theta \in \mathbb{R}^+ \;\middle|\; \begin{array}{ll} & (\Theta \leq \Delta) \\ \wedge & (\forall x \geq t : \alpha(x - t) + D_t \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t)) \end{array} \right\}. \tag{11}$$

Since sbf is not a continuous function, it is difficult to calculate $\Theta^*(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ in a straightforward linear way. Instead, we break the region below the sbf function into overlapping subregions we call *ℓ-feasibility regions* for any given $\Omega = (\Pi, \Theta, \Delta)$. An ℓ-feasibility region is the area below the ℓ^{th} "step" of the sbf function extending downward and infinitely to the right. Figure 3 gives a visual depiction; the definition below formalizes the ℓ-feasibility region concept.

Definition 5 (ℓ-Feasibility Region of Ω)

$$\mathscr{F}_\ell(\Pi, \Theta, \Delta) \stackrel{def}{=} \left\{ \langle (t, D_t), \alpha \rangle \in \mathbb{R}^2_{\geq 0} \times \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{ll} & \left(0 \leq \alpha \leq \frac{\Theta}{\Pi}\right) \\ \wedge & \left(\Theta \geq \frac{D_t - t + \ell\Pi + \Delta}{\ell + 1}\right) \\ \wedge & \left(\Theta \geq \frac{D_t}{\ell}\right) \\ \wedge & \left(\Theta \geq \frac{D_t + \alpha((\ell+1)\Pi + \Delta - t)}{\ell + 2\alpha}\right) \end{array} \right\}. \tag{12}$$

The next defined function determines the minimum capacity for any given half-line $\langle (t, D_t), \alpha \rangle$ to be an element of the ℓ-feasibility region.

Definition 6 (ℓ-Minimum Capacity for $\langle (t, D_t), \alpha \rangle$)

$$\Theta_\ell^*(\Pi, \Delta, \langle (t, D_t), \alpha \rangle) \stackrel{def}{=} \min \left\{ \Theta(\leq \Delta) \in \mathbb{R}^+ \mid \langle (t, D_t), \alpha \rangle \in \mathscr{F}_\ell(\Pi, \Theta, \Delta) \right\}. \tag{13}$$

We begin by showing that the half-line $\langle (t, D_t), \alpha \rangle$ is completely below sbf if and only if $\langle (t, D_t), \alpha \rangle$ is contained in some ℓ-feasibility region with $\ell > 0$.

Lemma 3 *For any $\langle (t, D_t), \alpha \rangle \in \mathbb{R}^2_{\geq 0} \times \mathbb{R}_{\geq 0}$, $\alpha(t' - t) + D_t \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t')\ \forall t' \geq t$ if and only if there exists $\ell \in \mathbb{N}$ such that $\langle (t, D_t), \alpha \rangle \in \mathscr{F}_\ell(\Pi, \Delta, \Theta)$.*

Proof Sketch: A complete algebraic proof is rather involved. Instead, we will give a geometric proof sketch. A more rigorous formal proof will appear in an extended version of this paper. We will first show the "if" direction. Assume that $\langle (t, D_t), \alpha \rangle \in \mathscr{F}_\ell(\Pi, \Delta, \Theta)$ for some $\ell \in \mathbb{N}$. We must show that the half-line $\langle (t, D_t), \alpha \rangle$ is completely contained below the sbf function for $\Omega = (\Pi, \Theta, \Delta)$. (This is equivalent to showing $\alpha(t' - t) + D_t \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t')\ \forall t' \geq t$). If $D_t \leq \mathsf{lsbf}((\Pi, \Theta, \Delta), t)$, then the half-line $\langle (t, D_t), \alpha \rangle$ is below $\mathsf{lsbf}(\Omega, t')$ for all $t' \geq t$, because the slope of the half-line (i.e., α) is at most $\frac{\Theta}{\Pi}$ (i.e., the slope of $\mathsf{lsbf}(\Omega, t)$). $\alpha \leq \frac{\Theta}{\Pi}$ follows from Equation 12). Since $\mathsf{lsbf}(\Omega, t') \leq \mathsf{sbf}(\Omega, t')$ for all $t' \geq t$, this implies that the half-line $\langle (t, D_t), \alpha \rangle$ never exceeds lsbf for $\Omega = (\Pi, \Theta, \Delta)$. Otherwise, if $D_t > \mathsf{lsbf}((\Pi, \Theta, \Delta), t)$, the originating point of the half-line $\langle (t, D_t), \alpha \rangle$ is contained within the triangular convex region defined by the (x, y)-points $(\ell\Pi + \Delta - 2\Theta, (\ell - 1)\Theta)$, $(\ell\Pi + \Delta - \Theta, \ell\Theta)$, and $((\ell + 1)\Pi + \Delta - 2\Theta, \ell\Theta)$; this region is formed by the intersection of the half-plane $y > \mathsf{lsbf}((\Pi, \Theta, \Delta), x)$ and the ℓ-feasibility region; e.g., see Figure 3 for an example of this intersection when $\ell = 2$. From the figure it is obvious that this triangular region is contained below sbf. Furthermore, the entire ℓ-feasibility region is contained below sbf; thus, $D_t \leq \mathsf{sbf}((\Pi, \Theta, \Delta), t)$. If we can show that the half-line $\langle (t, D_t), \alpha \rangle$'s value is below $\ell\Theta$ at $t_1 = (\ell + 1)\Pi + \Delta - 2\Theta$, then $[t, t_1]$ interval portion of the half-line is contained entirely within the ℓ-feasibility region (and, thus, below sbf), and the remaining $[t_1, \infty)$ portion of the half-line falls below lsbf (and, thus, sbf). From the fourth condition of Equation 12, it must be that $\ell\Theta \geq D_t + \alpha((\ell + 1)\Pi + \Delta - 2\Theta - t)$ which is equivalent to $\langle (t, D_t), \alpha \rangle$ being less than $\ell\Theta$ at t_1. Thus, in either case (based on the relative values of D_t and $\mathsf{lsbf}(\Omega, t)$), we show the half-line $\langle (t, D_t), \alpha \rangle$ must be completely contained below the sbf function for Ω.

For the "only if" direction, we must show that if the half-line $\langle (t, D_t), \alpha \rangle$ is completely contained below the sbf function for Ω, then there exists an $\ell \in \mathbb{N}$ such that $\langle (t, D_t), \alpha \rangle \in \mathscr{F}_\ell(\Pi, \Theta, \Delta)$. Consider $\ell = \lceil \frac{D_t}{\Theta} \rceil$. (If $\Theta = 0$, then we will simply use $\ell = 0$, since in this case D_t must be zero due to $\mathsf{sbf}((\Pi, 0, \Delta), t) = 0$ for all $t > 0$). The third condition of Equation 12 is trivially satisfied for this ℓ. It also must be true that $D_t > (\ell - 1)\Theta$. Thus, (t, D_t) must be below of the line defined by $y = x - (\ell\Pi + \Delta - (\ell + 1)\Theta)$ (otherwise, (t, D_t) would be above the sbf function at t). This last constraint is equivalent to the second condition of Equation 12. The half-line's slope α obviously must not exceed $\frac{\Theta}{\Pi}$; otherwise, the $\langle (t, D_t), \alpha \rangle$ would eventually exceed sbf at some point. This constraint corresponds to the first condition of Equation 12. Finally, $\ell\Theta$ must be greater than the value of $\langle (t, D_t), \alpha \rangle$ at $t_1 = (\ell + 1)\Pi + \Delta - 2\Theta$; otherwise, the half-line would exceed sbf at t_1. By the previous paragraph, we showed that this condition corresponds to satisfying the fourth condition of Equation 12. Therefore, for $\ell = \lceil \frac{D_t}{\Theta} \rceil$ we have satisfied all four conditions of Equation 12, implying that $\langle (t, D_t), \alpha \rangle \in \mathscr{F}_{\lceil \frac{D_t}{\Theta} \rceil}(\Pi, \Theta, \Delta)$. ■

Lemma 4 *For any $\ell \in \mathbb{N}^+$,*

$$\Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle) = \max \left\{ \begin{array}{r} \alpha\Pi, \\ \frac{D_t - t + \ell\Pi + \Delta}{\ell + 1}, \\ \frac{D_t}{\ell}, \\ \frac{D_t + \alpha((\ell+1)\Pi + \Delta - t)}{\ell + 2\alpha} \end{array} \right\}. \tag{14}$$

Proof: Let Θ_{RHS} denote the right-hand side of Equation 14. We will show that both $\Theta_{\mathrm{RHS}} \geq \Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ and $\Theta_{\mathrm{RHS}} \leq \Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ which will imply the lemma. The inequality $\Theta_{\mathrm{RHS}} \geq \Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ follows from the observation that $\langle (t, D_t), \alpha \rangle \in \mathscr{F}_\ell(\Pi, \Theta_{\mathrm{RHS}}, \Delta)$ (i.e., the four conditions of Equation 12 imply the lower bound).

We will show $\Theta_{\mathrm{RHS}} \leq \Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ by contradiction. Assume the negation of the inequality. In this case, since $\Theta_{\mathrm{RHS}} > \Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$, it must be that $\Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ is strictly less than at least one of the four terms in the RHS of Equation 14; however, this implies that $\langle (t, D_t), \alpha \rangle \notin \mathscr{F}_\ell(\Pi, \Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle), \Delta)$. The previous statement contradicts the definition of Definition 6; thus, it must be that $\Theta_{\mathrm{RHS}} \leq \Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ is true. ■

Now that we know how to efficiently compute $\Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ by simply checking the four values defined by the previous lemma, it would be convenient to use this value to compute the overall minimum capacity for half-line $\langle (t, D_t), \alpha \rangle$. The next lemma shows that this minimum bandwidth can be found by taking the minimum ℓ-minimum capacity for all positive ℓ.

Lemma 5

$$\Theta^*(\Pi, \Delta, \langle (t, D_t), \alpha \rangle) = \min_{\ell > 0} \left\{ \Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle) \right\}. \tag{15}$$

Proof: Let Θ_{RHS} denote the right-hand side of Equation 15. We will show that both $\Theta_{\mathrm{RHS}} \geq \Theta^*(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ and $\Theta_{\mathrm{RHS}} \leq \Theta^*(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ which will imply the lemma. First, we show $\Theta_{\mathrm{RHS}} \geq \Theta^*(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$. By Equation 15, Θ_{RHS} must equal $\Theta^*_\ell(\Pi, \Delta, \langle (t, D_t), \alpha \rangle)$ for some $\ell > 0$. Definition 6 states that $\langle (t, D_t), \alpha \rangle \in \mathscr{F}_\ell(\Pi, \Theta_{\mathrm{RHS}}, \Delta)$

for this ℓ. Lemma 3 then implies that $\alpha(t'-t)+D_t \leq \mathsf{sbf}((\Pi,\Theta_{\text{RHS}},\Delta),t')$ for all $t' \geq t$. Therefore, Θ_{RHS} must be in the set considered in the min on the right-hand side of Equation 11 in Definition 4. Thus, $\Theta_{\text{RHS}} \geq \Theta^*(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$.

Next, we will show $\Theta_{\text{RHS}} \leq \Theta^*(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$. By Definition 4,

$$\forall x \geq t : \alpha \cdot (x-t)+D_t \leq \mathsf{sbf}((\Pi,\Delta,\Theta^*(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)),t).$$

Lemma 3 implies that there exists $\ell \in \mathbb{N}$ such that $\langle(t,D_t),\alpha\rangle \in \mathscr{F}_\ell(\Pi,\Theta^*(\Pi,\Delta,\langle(t,D_t),\alpha\rangle),\Delta)$. By Definition 6, this implies that $\Theta^*(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$ is in the set considered in the right-hand side of Equation 13 which implies the inequality. ∎

Fortunately, we do not need to evaluate $\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$ for all positive ℓ to compute the minimum capacity for half-line $\langle(t,D_t),\alpha\rangle$. In the next lemma and corollary, we show that the smallest value of ℓ that needs to be considered is $\max\left(1,\left\lfloor\frac{t-\Delta}{\Pi}\right\rfloor\right)$ for a given Π and Δ. For all values of ℓ' smaller than this value, we can show that $\Theta^*_{\ell'}(\Pi,\Delta,\langle(t,D_t),\alpha\rangle) \geq \Theta^*_{\max\left(1,\left\lfloor\frac{t-\Delta}{\Pi}\right\rfloor\right)}(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$.

Lemma 6 *For any $\ell,\ell' \in \mathbb{N}^+$ and $D_t,t,\alpha \in \mathbb{R}_{\geq 0}$, if $(t \geq \ell\Pi+\Delta)$, $(\ell' < \ell)$, and $(\alpha \leq 1)$, then*

$$[\langle(t,D_t),\alpha\rangle \in \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)] \Rightarrow [\langle(t,D_t),\alpha\rangle \in \mathscr{F}_{\ell}(\Pi,\Theta,\Delta)].$$

Proof: Assume that $\langle(t,D_t),\alpha\rangle \in \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)$. We must show that all four conditions of Equation 12 are satisfied for $\mathscr{F}_\ell$. First note that $\langle(t,D_t),\alpha\rangle \in \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)$ implies that $\alpha \leq \frac{\Theta}{\Pi}$; thus, the first condition of Equation 5 is trivially satisfied for $\mathscr{F}_\ell$. By the third condition of Equation 12 for $\mathscr{F}_{\ell'}$, $D_t \leq \ell'\Theta$. Since $\ell > \ell'$, it must also be that $D_t \leq \ell\Theta$ (which satisfies third condition for $\mathscr{F}_\ell$). Similarly, the fourth condition of Equation 12 for $\mathscr{F}_{\ell'}$ implies that $\ell'\Theta \geq \alpha((\ell'+1)\Pi+\Delta-2\Theta-t)+D_t$. Since $\Theta \geq \alpha\Pi$ and $\ell \geq \ell'$, the fourth condition for $\mathscr{F}_\ell$ is also satisfied. Finally, consider the following expression

$$\begin{aligned}
&t-\ell\Pi-\Delta+(\ell+1)\Theta \\
&\quad \geq (\ell\Pi+\Delta)-(\ell\Pi+\Delta)+(\ell+1)\Theta \quad \text{(by assumption on } t\text{)} \\
&\quad = (\ell+1)\Theta \\
&\quad > \ell'\Theta \\
&\quad \geq D_t \quad \text{(from condition three for } \mathscr{F}_{\ell'}\text{)}
\end{aligned}$$

The above derivation implies the second condition for $\mathscr{F}_\ell$ which proves the lemma. ∎

Given t,Π, and Δ, consider $\ell = \max\left(1,\left\lfloor\frac{t-\Delta}{\Pi}\right\rfloor\right)$. For any $D_t,\alpha,\ell' < \ell$ which satisfy the supposition of the above lemma, it must be that.

$$\begin{aligned}
&\left\{\Theta(\leq\Delta) \in \mathbb{R}^+ \mid \langle(t,D_t),\alpha\rangle \in \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)\right\} \\
&\quad \subseteq \left\{\Theta(\leq\Delta) \in \mathbb{R}^+ \mid \langle(t,D_t),\alpha\rangle \in \mathscr{F}_{\max\left(1,\left\lfloor\frac{t-\Delta}{\Pi}\right\rfloor\right)}(\Pi,\Theta,\Delta)\right\}.
\end{aligned}$$

By the above expression and Equation 13 of Definition 6, the corollary below follows immediately.

Corollary 2 *For any $\ell' \in \mathbb{N}^+$ and $D_t,t,\alpha \in \mathbb{R}_{\geq 0}$, if $\left(\ell' < \max\left(1,\left\lfloor\frac{t-\Delta}{\Pi}\right\rfloor\right)\right)$ and $(\alpha \leq 1)$, then*

$$\Theta^*_{\ell'}(\Pi,\Delta,\langle(t,D_t),\alpha\rangle) \geq \Theta^*_{\max\left(1,\left\lfloor\frac{t-\Delta}{\Pi}\right\rfloor\right)}(\Pi,\Delta,\langle(t,D_t),\alpha\rangle).$$

In the following two lemmas and one corollary, we show that the largest value of ℓ that needs to be considered is $\left\lceil\frac{t+\Delta}{\Pi}\right\rceil-1$ for a given Π and Δ. For all values of ℓ' larger than this value, we can show that $\Theta^*_{\ell'}(\Pi,\Delta,\langle(t,D_t),\alpha\rangle) \geq \Theta^*_{\left\lceil\frac{t+\Delta}{\Pi}\right\rceil-1}(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$.

Lemma 7 *For any $\ell,\ell' \in \mathbb{N}^+$ and $D_t,t,\alpha \in \mathbb{R}_{\geq 0}$, if $(t \leq \ell\Pi-\Delta)$, $(\ell' \geq \ell-1)$, and $(\alpha \leq 1)$, then*

$$[\langle(t,D_t),\alpha\rangle \notin \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)] \Rightarrow [\langle(t,D_t),\alpha\rangle \notin \mathscr{F}_{\ell'+1}(\Pi,\Theta,\Delta)].$$

Proof: Assume that $\langle(t,D_t),\alpha\rangle \notin \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)$. This implies that at least one of the following is true:

$$\begin{array}{ll}
\alpha > \frac{\Theta}{\Pi} & (16a) \\
\Theta < \frac{D_t-t+\ell'\Pi+\Delta}{\ell'+1} & (16b) \\
\Theta < \frac{D_t}{\ell'} & (16c) \\
\Theta < \frac{D_t+\alpha((\ell'+1)\Pi+\Delta-t)}{\ell'+2\alpha} & (16d)
\end{array} \qquad (16)$$

We will show that each of the above inequalities implies that $\langle(t,D_t),\alpha\rangle \notin \mathscr{F}_{\ell'+1}(\Pi,\Theta,\Delta)$. Obviously, Equation 16a trivially implies the aforementioned expression. If Equation 16b is true, then

$$\begin{aligned}
&D_t > t-\ell'\Pi-\Delta+(\ell'+1)\Theta \\
&\quad \Rightarrow D_t > t-(\ell'+1)\Pi-\Delta+(\ell'+2)\Theta \\
&\quad \Rightarrow \Theta > \frac{D_t-t+(\ell'+1)\Pi+\Delta}{\ell'+2}
\end{aligned}$$

The first implication above is due to the fact that right-hand side of the first inequality in the derivation is non-increasing in ℓ' (since $\Theta \leq \Pi$). Thus, if Equation 16b is true, then $\langle(t,D_t),\alpha\rangle \notin \mathscr{F}_{\ell'+1}(\Pi,\Theta,\Delta)$.

If Equation 16c is true, then consider the following expression:

$$\begin{aligned}
&t-(\ell'+1)\Pi-\Delta+(\ell'+2)\Theta \\
&\quad \leq \ell\Pi-\Delta-(\ell'+1)\Pi-\Delta+(\ell'+2)\Theta \\
&\quad\quad\quad \text{(by supposition on } t\text{)} \\
&\quad \leq \ell\Pi-\ell\Pi-2\Delta+(\ell'+2)\Theta \quad \text{(by } \ell' \geq \ell-1\text{)} \\
&\quad \leq (\ell'+2)\Theta-2\Theta \quad \text{(by } \Delta \geq \Theta\text{)} \\
&\quad = \ell'\Theta \\
&\quad < D_t \quad \text{(by Equation 16c)}
\end{aligned}$$

Thus, $\Theta < \frac{D_t-t+(\ell'+1)\Pi+\Delta}{\ell'+2}$ which implies $\langle(t,D_t),\alpha\rangle \notin \mathscr{F}_{\ell'+1}(\Pi,\Theta,\Delta)$.

Finally, if Equation 16d is true, then

$$\begin{aligned}&\ell'\Theta < D_t + \alpha((\ell'+1)\Pi + \Delta - 2\Theta - t)\\ &\Rightarrow \ell'\Theta < D_t + (\ell'+1)\Pi + \Delta - 2\Theta - t\\ &\Rightarrow (\ell'+2)\Theta < D_t + (\ell'+1)\Pi + \Delta - t\\ &\Rightarrow \Theta < \tfrac{D_t - t + (\ell'+1)\Pi + \Delta}{\ell'+2}\end{aligned}$$

The first implication is due to $\alpha \leq 1$. The last inequality implies that when Equation 16d is true, $\langle (t, D_t), \alpha\rangle \notin \mathscr{F}_{\ell'+1}(\Pi,\Theta,\Delta)$; the lemma follows. ■

Lemma 8 *For any $\ell, \ell' \in \mathbb{N}^+$ and $D_t, t, \alpha \in \mathbb{R}_{\geq 0}$, if $(t \leq \ell\Pi - \Delta)$, $(\ell' > \ell - 1)$, and $(\alpha \leq 1)$, then*

$$[\langle (t, D_t), \alpha\rangle \in \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)] \Rightarrow [\langle (t, D_t), \alpha\rangle \in \mathscr{F}_{\ell-1}(\Pi,\Theta,\Delta)].$$

Proof: By the contrapositive of Lemma 7, if $\langle (t, D_t), \alpha\rangle \in \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)$, then $\langle (t, D_t), \alpha\rangle \in \mathscr{F}_{\ell'-1}(\Pi,\Theta,\Delta)$. If $\ell' - 1$ equals $\ell - 1$, then we have shown the lemma. Otherwise, the lemma follows from repeated application of the contrapositive of Lemma 7. ■

Given t, Π, and Δ, consider $\ell = \lceil \frac{t+\Delta}{\Pi} \rceil$. For any $D_t, \alpha, \ell' > \ell - 1$ which satisfy the supposition of the above lemma, it must be that.

$$\begin{aligned}&\left\{\Theta(\leq \Delta) \in \mathbb{R}^+ \mid \langle (t, D_t), \alpha\rangle \in \mathscr{F}_{\ell'}(\Pi,\Theta,\Delta)\right\}\\ &\quad \subseteq \left\{\Theta(\leq \Delta) \in \mathbb{R}^+ \mid \langle (t, D_t), \alpha\rangle \in \mathscr{F}_{\lceil \frac{t+\Delta}{\Pi} \rceil - 1}(\Pi,\Theta,\Delta)\right\}.\end{aligned}$$

By the above expression and Equation 13 of Definition 6, the corollary below follows immediately.

Corollary 3 *For a given Π, Δ, and $\langle (t, D_t), \alpha\rangle$, the following is true for all $\ell' > \lceil \frac{t+\Delta}{\Pi} \rceil - 1$,*

$$\Theta^*_{\ell'}(\Pi,\Delta,\langle (t, D_t), \alpha\rangle) \geq \Theta^*_{\lceil \frac{t+\Delta}{\Pi} \rceil - 1}(\Pi,\Delta,\langle (t, D_t), \alpha\rangle).$$

Combining Corollaries 2 and 3 with Lemma 5, we obtain the following lemma which show that $\Theta^*(\Pi,\Delta,\langle (t, D_t), \alpha\rangle)$ may be calculated with only a small number of values for ℓ.

Corollary 4

$$\begin{aligned}&\Theta^*(\Pi,\Delta,\langle (t, D_t), \alpha\rangle) =\\ &\quad \min_{\max\left\{1, \left(\lfloor \frac{t-\Delta}{\Pi} \rfloor\right)\right\} \leq \ell \leq \left(\lceil \frac{t+\Delta}{\Pi} \rceil - 1\right)} \left\{\Theta^*_\ell(\Pi,\Delta,\langle (t, D_t), \alpha\rangle)\right\}.\end{aligned}$$

The next lemma shows the maximum value of Θ obtained from calculating $\Theta^*(\cdot)$ for every value in the testing set $\widetilde{\mathsf{TS}}(\tau,k)$ may be safely used as a capacity which will satisfy the condition of Theorem 1 (i.e., the EDP schedulability condition).

Lemma 9 *For any $t \geq 0$ $\widetilde{\mathsf{DBF}}(\tau,t,k) \leq \mathsf{sbf}((\Pi,\Theta,\Delta),t)$ and $U(\tau) \leq \frac{\Theta}{\Pi}$, if and only if,*

$$\Theta \geq \max\left(\begin{array}{l}\max_{t \in \widetilde{\mathsf{TS}}(\tau,k)} \left\{\Theta^*\left(\Pi,\Delta,\langle (t, \widetilde{\mathsf{DBF}}(\tau,t,k)), \psi(\tau,t,k)\rangle\right)\right\},\\ U(\tau)\cdot\Pi\end{array}\right). \tag{17}$$

Proof: We will show the "if" direction first, by showing its contrapositive. We must show that if either $U(\tau) > \frac{\Theta}{\Pi}$ or $\exists t \geq 0 : \widetilde{\mathsf{DBF}}(\tau,t,k) > \mathsf{sbf}((\Pi,\Theta,\Delta),t)$, then the negation of the inequality of Equation 17 is also true. Assume that $U(\tau) > \frac{\Theta}{\Pi}$. By the second expression in the outer "max" of Equation 17, the negation of the inequality is true. Now, assume there exists $t \geq 0$ such that $\widetilde{\mathsf{DBF}}(\tau,t,k) > \mathsf{sbf}((\Pi,\Theta,\Delta),t)$. There must exist two consecutive values t_a and t_{a+1} in $\widetilde{\mathsf{TS}}(\tau,k)$ where $t \in [t_a, t_{a+1})$. (Recall that if t_a is the largest value in $\widetilde{\mathsf{TS}}(\tau,k)$ then t_{a+1} is assumed to be ∞). Lemma 2 implies that for all $t' \in [t_a, t_{a+1})$, $\widetilde{\mathsf{DBF}}(\tau,t',k)$ equals $\psi(\tau,t_a,k)\cdot(t'-t_a) + \widetilde{\mathsf{DBF}}(\tau,t_a,k)$. Since $t \in [t_a, t_{a+1})$,

$$\mathsf{sbf}((\Pi,\Theta,\Delta),t) < \psi(\tau,t_a,k)\cdot(t-t_a) + \widetilde{\mathsf{DBF}}(\tau,t_a,k).$$

By Lemma 3, this implies that for all $\ell \in \mathbb{N}$,

$$\langle (t_a, \widetilde{\mathsf{DBF}}(\tau,t_a,k)), \psi(\tau,t_a,k)\rangle \notin \mathscr{F}_\ell(\Pi,\Theta,\Delta).$$

By Equation 13 from Definition 6,

$$\Theta < \Theta^*_\ell(\Pi,\Delta,\langle (t_a, \widetilde{\mathsf{DBF}}(\tau,t_a,k)), \psi(\tau,t_a,k)\rangle)$$

for all $\ell \in \mathbb{N}$, which implies (by Lemma 5) $\Theta < \Theta^*(\Pi,\Delta,\langle (t_a, \widetilde{\mathsf{DBF}}(\tau,t_a,k)), \psi(\tau,t_a,k)\rangle)$. Thus, $\Theta < \max_{t \in \widetilde{\mathsf{TS}}(\tau,k)} \left\{\Theta^*\left(\Pi,\Delta,\langle (t, \widetilde{\mathsf{DBF}}(\tau,t,k)), \psi(\tau,t,k)\rangle\right)\right\}$. Again, the negation of the inequality of Equation 17 is true. The contrapositive of the "if" direction follows.

For the "only if" direction of the lemma, we will also consider the contrapositive. The contrapositive will follow by simply reversing the implications of the proof for the "if" direction. We give the argument for completeness. Assume that the negation of the inequality of Equation 17. If $\Theta < U(\tau)\cdot\Pi$, then obviously $U(\tau) > \frac{\Theta}{\Pi}$. Otherwise, if

$$\Theta < \max_{t \in \widetilde{\mathsf{TS}}(\tau,k)} \left\{\Theta^*\left(\Pi,\Delta,\langle (t, \widetilde{\mathsf{DBF}}(\tau,t,k)), \psi(\tau,t,k)\rangle\right)\right\},$$

then there exists $t_a \in \widetilde{\mathsf{TS}}(\tau,k)$ such that $\Theta < \Theta^*(\Pi,\Delta,\langle (t_a, \widetilde{\mathsf{DBF}}(\tau,t_a,k)), \psi(\tau,t_a,k)\rangle)$. By Lemma 5,

$$\Theta < \Theta^*_\ell(\Pi,\Delta,\langle (t_a, \widetilde{\mathsf{DBF}}(\tau,t_a,k)), \psi(\tau,t_a,k)\rangle)$$

for all $\ell \in \mathbb{N}$. From Equation 13 from Definition 6, $\langle (t_a, \widetilde{\mathsf{DBF}}(\tau,t_a,k)), \psi(\tau,t_a,k)\rangle \notin \mathscr{F}_\ell(\Pi,\Theta,\Delta)$ for all $\ell \in \mathbb{N}$. By Lemma 3, there exists a $t \geq t_a$ such that $\mathsf{sbf}((\Pi,\Theta,\Delta),t) < \psi(\tau,t_a,k)\cdot(t-t_a) + \widetilde{\mathsf{DBF}}(\tau,t_a,k)$. Lemma 2 implies that for all $t' \geq t_a$, $\widetilde{\mathsf{DBF}}(\tau,t',k)$ is at least $\psi(\tau,t_a,k)\cdot(t'-t_a) + \widetilde{\mathsf{DBF}}(\tau,t_a,k)$. Thus, since $t \geq t_a$, $\mathsf{sbf}((\Pi,\Theta,\Delta),t) < \widetilde{\mathsf{DBF}}(\tau,t,k)$. The contrapositive of the "only if" direction follows. ■

After proving the above lemma, we may now prove Theorem 2 which states that MINIMUMCAPACITY returns a valid value for finite k and an exact value for $k=\infty$.

Proof of Theorem 2 It is easy to see that $\Theta^{\min}$ returned from MINIMUMCAPACITY corresponds to the value on the right-hand side of Equation 17 of Lemma 9; the loop from Line 2 to 11 iterate through each value t_a in $\widetilde{\mathsf{TS}}(\tau,k)$ (setting variables corresponding to $\widetilde{\mathsf{DBF}}(\tau,t_a,k)$ and $\psi(\tau,t_a,k)$ in Lines 3 and 4, respectively). The inner loop from Line 6 to 9 determines $\Theta^*\left(\Pi,\Delta,\langle(t_a,\widetilde{\mathsf{DBF}}(\tau,t_a,k)),\psi(\tau,t_a,k)\rangle\right)$; the implications of Corollary 4 and Lemma 5 show that we need calculate $\Theta^*_\ell(\Pi,\Delta,\langle(t_a,\widetilde{\mathsf{DBF}}(\tau,t_a,k)),\psi(\tau,t_a,k)\rangle)$ only for values of ℓ from $\max\left(1,\left\lfloor\frac{t_a-\Delta}{\Pi}\right\rfloor\right)$ to $\left\lceil\frac{t_a+\Delta}{\Pi}\right\rceil-1$. Finally, in Line 10, $\Theta^{\min}$ is set to the maximum of $U(\tau)\cdot\Pi$ and $\Theta^*\left(\Pi,\Delta,\langle(t_a,\widetilde{\mathsf{DBF}}(\tau,t_a,k)),\psi(\tau,t_a,k)\rangle\right)$ over all values t_a in $\widetilde{\mathsf{TS}}(\tau,k)$.

By Lemma 9, $\widetilde{\mathsf{DBF}}(\tau,t,k)\le\mathsf{sbf}((\Pi,\Theta^{\min},\Delta),t)$ for all $t\ge0$ and $U(\tau)\le\frac{\Theta}{\Pi}$. By Lemma 1, $\mathsf{DBF}(\tau,t)\le\widetilde{\mathsf{DBF}}(\tau,t,k)\le\mathsf{sbf}((\Pi,\Theta^{\min},\Delta),t)$ for all $t\ge0$. Therefore, the supposition of Theorem 1 are satisfied and the τ will always meet all deadlines when scheduled by EDF upon $\Omega=(\Pi,\Theta^{\min},\Delta)$. When $k=\infty$, $\widetilde{\mathsf{DBF}}(\tau,t,k)$ equals $\mathsf{DBF}(\tau,t)$ for all $t\ge0$; in this case, $\Theta^{\min}$ equals $\Theta^*(\Pi,\Delta,\tau)$ (i.e., $\Theta^{\min}$ is an exact value) due to the fact that both Lemma 9 and Theorem 1 are necessary and sufficient. ■

5 An Approximation Scheme

In the previous section, we have shown that MINIMUMCAPACITY gives a valid answer when k is finite and an exact answer when k is infinite. When k is finite, we have not, yet, given any details on how accurate the returned $\Theta^{\min}$ will be. In this section, we show that we may trade computational efficiency for accuracy; that is, as k increases the guaranteed accuracy of MINIMUMCAPACITY increases along with its running time. Theorem 3 quantifies this tradeoff for a given k; Corollary 6 shows that this tradeoff permits an FPTAS for the MIB-RT problem. Before we prove Theorem 3 and Corollary 6, we need to prove two technical lemmas.

Lemma 10 *Given Π, Δ, and τ, the following is true for all $k,\ell(\in\mathbb{N}^+)$, $t,D_t(\in\mathbb{R}_{>0})$, and $\alpha(\in[0,1])$,*

$$\begin{aligned}&\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)\\ &\quad\le\left(\tfrac{k+1}{k}\right)\cdot\Theta^*_\ell\left(\Pi,\Delta,\left\langle\left(t,\tfrac{k\cdot D_t}{k+1}\right),\tfrac{k\cdot\alpha}{k+1}\right\rangle\right).\end{aligned}\tag{18}$$

Proof: By Lemma 4, $\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$ must be equal to one of the following: $\alpha\Pi$; $\frac{D_t-t+\ell\Pi+\Delta}{\ell+1}$; $\frac{D_t}{\ell}$; or $\frac{D_t+\alpha((\ell+1)\Pi+\Delta-t)}{\ell+2\alpha}$. We will show that for each of the four possibilities, Equation 18 must hold. If $\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$ is equal to $\alpha\Pi$, then by Lemma 4,

$$\begin{aligned}&\Theta^*_\ell\left(\Pi,\Delta,\left\langle\left(t,\frac{k\cdot D_t}{k+1}\right),\frac{k\cdot\alpha}{k+1}\right\rangle\right)\\ &\ge\left(\tfrac{k\cdot\alpha}{k+1}\right)\cdot\Pi\\ &=\left(\tfrac{k}{k+1}\right)\cdot\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle).\end{aligned}$$

The last step implies Equation 18.

If $\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$ is equal to $\frac{D_t-t+\ell\Pi+\Delta}{\ell+1}$, we will consider two subcases: $t\le\ell\Pi+\Delta$ and $t>\ell\Pi+\Delta$. If $t\le\ell\Pi+\Delta$, then by Lemma 4,

$$\begin{aligned}&\Theta^*_\ell\left(\Pi,\Delta,\left\langle\left(t,\frac{k\cdot D_t}{k+1}\right),\frac{k\cdot\alpha}{k+1}\right\rangle\right)\\ &\ge\tfrac{\frac{k\cdot D_t}{k+1}-t+\ell\Pi+\Delta}{\ell+1}\\ &\ge\left(\tfrac{k}{k+1}\right)\cdot\left(\tfrac{D_t-t+\ell\Pi+\Delta}{\ell+1}\right)\\ &=\left(\tfrac{k}{k+1}\right)\cdot\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle).\end{aligned}$$

Otherwise, $t>\ell\Pi+\Delta$. In this case, Lemma 4 also implies

$$\begin{aligned}&\Theta^*_\ell\left(\Pi,\Delta,\left\langle\left(t,\frac{k\cdot D_t}{k+1}\right),\frac{k\cdot\alpha}{k+1}\right\rangle\right)\\ &\ge\tfrac{\frac{k\cdot D_t}{k+1}}{\ell}\\ &\ge\tfrac{\left(\frac{k}{k+1}\right)\cdot D_t-\left(\frac{k}{k+1}\right)(t-\ell\Pi+\Delta)}{\ell+1}\\ &=\left(\tfrac{k}{k+1}\right)\cdot\left(\tfrac{D_t-t+\ell\Pi+\Delta}{\ell+1}\right)\\ &=\left(\tfrac{k}{k+1}\right)\cdot\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle).\end{aligned}$$

Thus, in either subcase, Equation 18 holds.

If $\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$ is equal to $\frac{D_t}{\ell}$, then Lemma 4 implies

$$\begin{aligned}&\Theta^*_\ell\left(\Pi,\Delta,\left\langle\left(t,\frac{k\cdot D_t}{k+1}\right),\frac{k\cdot\alpha}{k+1}\right\rangle\right)\\ &\ge\tfrac{\frac{k\cdot D_t}{k+1}}{\ell}\\ &=\left(\tfrac{k}{k+1}\right)\cdot\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle).\end{aligned}$$

Finally, if $\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle)$ is equal to $\frac{D_t+\alpha((\ell+1)\Pi+\Delta-t)}{\ell+2\alpha}$, then Lemma 4 implies that

$$\begin{aligned}&\Theta^*_\ell\left(\Pi,\Delta,\left\langle\left(t,\frac{k\cdot D_t}{k+1}\right),\frac{k\cdot\alpha}{k+1}\right\rangle\right)\\ &\ge\tfrac{\frac{k\cdot D_t}{k+1}-\left(\frac{k\cdot\alpha}{k+1}\right)((\ell+1)\Pi+\Delta-t)}{\ell+2\cdot\left(\frac{k\cdot\alpha}{k+1}\right)}\\ &\ge\left(\tfrac{k}{k+1}\right)\cdot\left(\tfrac{D_t-\alpha((\ell+1)\Pi+\Delta-t)}{\ell+2\alpha}\right)\\ &=\left(\tfrac{k}{k+1}\right)\cdot\Theta^*_\ell(\Pi,\Delta,\langle(t,D_t),\alpha\rangle).\end{aligned}$$

■

Lemma 11 *Given Π, Δ, and τ, the following is true for all $k\in\mathbb{N}^+$ and $t_a\in\widetilde{\mathsf{TS}}(\tau,k)$,*

$$\Theta^*(\Pi,\Delta,\tau)\ge\Theta^*\left(\Pi,\Delta,\left\langle\left(t_a,\frac{k\cdot\widetilde{\mathsf{DBF}}(\tau,t_a,k)}{k+1}\right),\frac{k\cdot\psi(\tau,t_a,k)}{k+1}\right\rangle\right).\tag{19}$$

Proof: Let Θ_{RHS} denote the right-hand side of Equation 19. By definition of $\Theta^*(\Pi,\Delta,\tau)$ and Theorem 1,

$$\mathsf{sbf}((\Pi,\Theta^*(\Pi,\Delta,\tau),\Delta),t) \geq \mathsf{DBF}(\tau,t) : \forall t \geq 0. \quad (20)$$

Now consider any $t_a \in \widetilde{\mathsf{TS}}(\tau,k)$. By combining Lemma 1 and Lemma 2, we have, for all $t \geq t_a$,

$$\left(\frac{k+1}{k}\right) \cdot \mathsf{DBF}(\tau,t) \geq \psi(\tau,t_a,k)\cdot(t-t_a) + \widetilde{\mathsf{DBF}}(\tau,t_a,k). \quad (21)$$

Combining the inequalities of Equations 20 and 21 gives us, for all $t \geq t_a$,

$$\begin{aligned}&\mathsf{sbf}((\Pi,\Theta^*(\Pi,\Delta,\tau),\Delta),t)\\ &\quad\geq \left(\tfrac{k\cdot\psi(\tau,t_a,k)}{k+1}\right)\cdot(t-t_a) + \tfrac{k\cdot\widetilde{\mathsf{DBF}}(\tau,t_a,k)}{k+1}.\end{aligned} \quad (22)$$

Lemma 3 and Equation 22 imply that there exists $\ell \in \mathbb{N}$ such that

$$\left\langle \left(t_a, \frac{k\cdot\widetilde{\mathsf{DBF}}(\tau,t_a,k)}{k+1}\right), \frac{k\cdot\psi(\tau,t_a,k)}{k+1}\right\rangle \in \mathscr{F}_\ell(\Pi,\Theta^*(\Pi,\Delta,\tau),\Delta).$$

The above expression and Definition 6 imply

$$\Theta^*_\ell\left(\Pi,\Delta,\left\langle\left(t_a,\frac{k\cdot\widetilde{\mathsf{DBF}}(\tau,t_a,k)}{k+1}\right),\frac{k\cdot\psi(\tau,t_a,k)}{k+1}\right\rangle\right) \leq \Theta^*(\Pi,\Delta,\tau).$$

The lemma follows from the expression above and Lemma 5. ■

The following corollary gives upper and lower bounds on the value obtained from the approximation. The corollary follows by combining Lemmas 10 and 11.

Corollary 5 *Given* Π, Δ, *and* τ, *the following is true for all* $k \in \mathbb{N}^+$ *and* $t \in \widetilde{\mathsf{TS}}(\tau,k)$,

$$\begin{aligned}&\left(\frac{k+1}{k}\right)\cdot\Theta^*(\Pi,\Delta,\tau)\\ &\quad\geq \min_{\ell\in\mathbb{N}^+}\left\{\Theta^*_\ell\left(\Pi,\Delta,\left\langle\left(t,\widetilde{\mathsf{DBF}}(\tau,t,k)\right),\psi(\tau,t,k)\right\rangle\right)\right\}.\end{aligned} \quad (23)$$

Finally, we give the theorem which quantifies the trade-off between accuracy and computational complexity, in terms of k.

Theorem 3 *Given* Π, Δ, τ, *and* $k \in \mathbb{N}^+$, *the procedure* MINIMUMCAPACITY *returns* $\Theta^{\min}$ *such that*

$$\Theta^*(\Pi,\Delta,\tau) \leq \Theta^{\min} \leq \left(\frac{k+1}{k}\right)\cdot\Theta^*(\Pi,\Delta,\tau).$$

Furthermore, MINIMUMCAPACITY (Π,Δ,τ,k) *has time complexity* $O(kn\lg n)$

Proof: Theorem 2 shows that $\Theta^*(\Pi,\Delta,\tau) \leq \Theta^{\min}$; thus, we are left to prove the second inequality of the theorem. There are two cases that we will consider: $\Theta^{\min} = U(\tau)\cdot\Pi$ and $\Theta^{\min} > U(\tau)\cdot\Pi$. (Observe by Line 1, $\Theta^{\min}$ must be at least $U(\tau)\cdot\Pi$). In the case that $\Theta^{\min}$ equals $U(\tau)\cdot\Pi$, Theorem 1 implies that $\Theta^*(\Pi,\Delta,\tau)$ must be at least $U(\tau)\cdot\Pi$. For this case, the second inequality follows, since $\frac{k+1}{k} \geq 1$ for all $k \in \mathbb{N}^+$.

In the case that $\Theta^{\min}$ exceeds $U(\tau)\cdot\Pi$, $\Theta^{\min}$ must equal

$$\max_{t\in\widetilde{\mathsf{TS}}(\tau,k)}\left\{\Theta^*\left(\Pi,\Delta,\langle(t,\widetilde{\mathsf{DBF}}(\tau,t,k)),\psi(\tau,t,k)\rangle\right)\right\}$$

according to the proof of Theorem 2. By Lemma 5, this is equivalent to

$$\max_{t\in\widetilde{\mathsf{TS}}(\tau,k)}\left\{\min_{\ell\in\mathbb{N}^+}\left\{\Theta^*_\ell\left(\Pi,\Delta,\langle(t,\widetilde{\mathsf{DBF}}(\tau,t,k)),\psi(\tau,t,k)\rangle\right)\right\}\right\}.$$

Applying Corollary 5 immediately yields the second inequality of the theorem for this case. ■

If we are given an accuracy parameter $\varepsilon > 0$, we can appropriately set k equal to $\lceil\frac{1}{\varepsilon}\rceil$ and guarantee an $(1+\varepsilon)$ approximation ratio from the result returned from MINIMUMCAPACITY. The following corollary (which follows from Theorem 3) shows that this approach yields an FPTAS for MIB-RT.

Corollary 6 *Given* Π, Δ, τ, *and* $\varepsilon > 0$, *the procedure* MINIMUMCAPACITY $\left(\Pi,\Delta,\tau,\lceil\frac{1}{\varepsilon}\rceil\right)$ *returns* $\Theta^{\min}$ *such that*

$$\Theta^*(\Pi,\Delta,\tau) \leq \Theta^{\min} \leq (1+\varepsilon)\cdot\Theta^*(\Pi,\Delta,\tau).$$

Furthermore, MINIMUMCAPACITY $\left(\Pi,\Delta,\tau,\lceil\frac{1}{\varepsilon}\rceil\right)$ *has time complexity* $O\left(\frac{n\lg n}{\varepsilon}\right)$.

6 Simulations

In this section, we show the simulation results for our proposed algorithm and compare it with the exact [10] and sufficient algorithm [21, 12]. During simulations, we have the following simulation parameters and value ranges (based on the methodology of [21]):

1. The number of tasks in a task system τ is $2,4,8,16$, or 32.
2. The system utilization $U(\tau)$ is taken from the range $[0.1,0.8]$ at 0.05-increments and individual task utilizations u_i are generated using UUniFast algorithm [7].
3. Each sporadic task $\tau = (e_i,d_i,p_i)$ has a period p_i uniformly drawn from the interval $[5,20]$. (A small period range is used to keep $P(\tau)$ from becoming too large). The execution time requirement e_i set to $u_i.p_i$. For each task, d_i equals p_i.

4. The component level scheduling algorithm is EDF.

5. The value of k is set to $3, 4$, or 5; Π is set to $5, 10$, or 15; Δ is equal to Π.

For each simulation, given task system size n and system utilization $U(\tau)$, we randomly generate taskset parameters u_i, p_i, and e_i for each task τ_i. We execute the linear time algorithm [21], the exact algorithm [10, 21] and MINIMUMCAPACITY to generate sufficient, exact and approximate capacity, respectively. Each point in the following plots represents the mean of 1000 simulation results. For this paper, due to space considerations, we show the result for $n = 8, k = 3, \Pi = 5$ only. Results for other parameters will be included in an extended version of this paper.

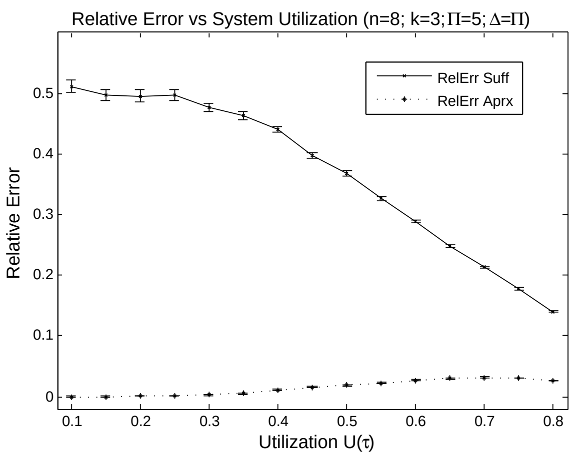

Figure 4. Relative Error vs System Utilization

In Figure 4, relative error in calculation of capacity for the two algorithms are plotted as a function of task system utilization. Relative error is defined as follows: $\frac{\Theta - \Theta^*}{\Theta^*}$. In this case, the estimated capacity Θ is either the sufficient capacity (denoted by $\widehat{\Theta}$) or approximate capacity (denoted by $\Theta^{\min}$). In the graph, the solid-line curve represents relative error for $\widehat{\Theta}$ and the dotted-line curve represents relative error for $\Theta^{\min}$. For MINIMUMCAPACITY, the mean relative error is less than 5%, whereas for the sufficient algorithm it ranges from 15% to 50%. The 95% confidence intervals are shown. Please note that approximation ratio for $\widehat{\Theta}$ is between $3/2$ and 3 as shown in [12] whereas the approximation ratio for $\Theta^{\min}$ is at most $4/3$ when $k = 3$ (from Theorem 3).

As we have mentioned before, the runtime complexity of MINIMUMCAPACITY entirely depends on the size of the testing set. Figure 5 shows a logarithmic plot to compare between testing set sizes for exact algorithm ($|\mathsf{TS}(\tau)|$) and approximate algorithm ($|\widetilde{\mathsf{TS}}(\tau, k)|$). The solid-line curve in the graph represents $|\widetilde{\mathsf{TS}}(\tau, k)|$ and the dotted-line

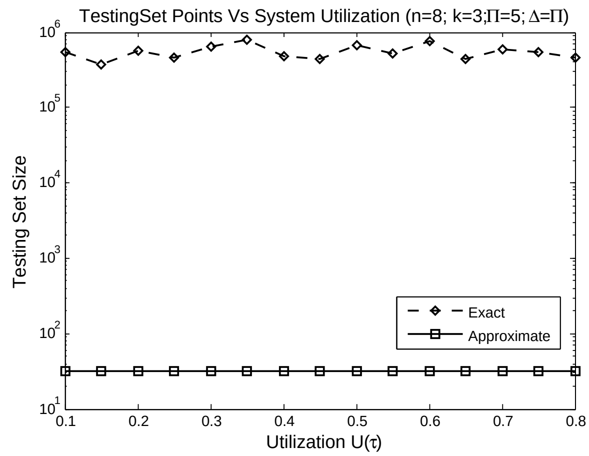

Figure 5. Testing Set size vs System Utilization

curve represents $|\mathsf{TS}(\tau)|$. As we know from the algorithm, $|\widetilde{\mathsf{TS}}(\tau, k)|$ depends on only the input k and taskset size n (which is constant for our graph), on the other hand $|\mathsf{TS}(\tau)|$ may be exponentially large and has several orders of magnitude of variance, since it depends on the lcm of the periods. The sufficient algorithm is not shown since the algorithm only uses the utilization and minimum period parameters (i.e., it is linear time). The simulation results strengthen the claim that MINIMUMCAPACITY improves upon the performance of the sufficient algorithm while still maintaining a low polynomial runtime.

7 Conclusions and Future Work

In this paper, we have proposed an approximation algorithm for the minimization of interface bandwidth (MIB-RT) problem in a real-time compositional framework, the explicit-deadline periodic (EDP) resource model. For this model and any sporadic task system, our algorithm returns bandwidth that is at most a factor of $(1 + \varepsilon)$ greater than the optimal minimum bandwidth, for any $\varepsilon > 0$. Furthermore, it is shown that our algorithm is an FPTAS as it has time complexity that is polynomial in the number of tasks in the sporadic task system and the term $1/\varepsilon$. It has been previously shown that currently-known polynomial-time algorithms have constant-factor approximation ratios, but cannot guarantee an approximation closer than a factor of $\frac{3}{2}$ larger than optimal. Furthermore, previous work [21, 10] has shown that exact algorithms for MIB-RT problem on periodic resources may require exponential time. Simulation results have shown that our approximation algorithm is effective at reducing the relative error over synthetically generated tasks, while maintaining a low runtime complexity.

We believe that the attainment of parametric approximation algorithms for MIB-RT problem under a variety compositional frameworks will provide a real-time component designer with a valuable choice in determining how much interface bandwidth to trade for decreased speed-of-analysis. Our goal is to extend the work contained in this paper to more general task models (e.g., generalized multiframe tasks [4]). Furthermore, we expect that parametric approximation algorithms for MIB-RT on uniprocessor frameworks will also extend to multiprocessor compositional frameworks (e.g., [18]).

References

[1] K. Albers, F. Bodmann, and F. Slomka. Advanced hierarchical event-stream model. In *Proceedings of the EuroMicro Conference on Real-Time Systems*, pages 211–220, Prague, Czech Republic, July 2008. IEEE Computer Society.

[2] K. Albers and F. Slomka. An event stream driven approximation for the analysis of real-time systems. In *Proceedings of the EuroMicro Conference on Real-Time Systems*, pages 187–195, Catania, Sicily, July 2004. IEEE Computer Society Press.

[3] L. Almeida and P. Pedreiras. Scheduling within temporal partitions: response-time analysis and server design. In *EMSOFT '04: Proceedings of the 4th ACM international conference on Embedded software*, pages 95–103, New York, NY, USA, 2004. ACM.

[4] S. Baruah, D. Chen, S. Gorinsky, and A. Mok. Generalized multiframe tasks. *Real-Time Systems: The International Journal of Time-Critical Computing*, 17(1):5–22, July 1999.

[5] S. Baruah, R. Howell, and L. Rosier. Feasibility problems for recurring tasks on one processor. *Theoretical Computer Science*, 118(1):3–20, 1993.

[6] S. Baruah, A. Mok, and L. Rosier. Preemptively scheduling hard-real-time sporadic tasks on one processor. In *Proceedings of the 11th Real-Time Systems Symposium*, pages 182–190, Orlando, Florida, 1990. IEEE Computer Society Press.

[7] E. Bini and G. Buttazzo. Biasing effects in schedulability measures. In *Proceedings of the 16th Euromicro Conference on Real-Time Systems*, pages 196–203. IEEE Computer Society, 2004.

[8] S. Chakraborty, S. Kunzli, and L. Thiele. A general framework for analysing system properties in platform-based embedded system designs. In *DATE '03: Proceedings of the conference on Design, Automation and Test in Europe*, page 10190, Washington, DC, USA, 2003. IEEE Computer Society.

[9] Z. Deng and J. Liu. Scheduling real-time applications in an Open environment. In *Proceedings of the Eighteenth Real-Time Systems Symposium*, pages 308–319, San Francisco, CA, December 1997. IEEE Computer Society Press.

[10] A. Easwaran, M. Anand, and I. Lee. Compositional analysis framework using EDP resource models. In *Proceedings of the IEEE Real-time Systems Symposium*, Tuscon, Arizona, December 2007. IEEE Computer Society.

[11] X. A. Feng and A. Mok. A model of hierarchical real-time virtual resources. In *Proceedings of the IEEE Real-Time Systems Symposium*, pages 26–35. IEEE Computer Society, 2002.

[12] N. Fisher. Approximation algorithms for compositional real-time systems: Trading bandwidth for speed-of-analysis. In *Proceedings of the Workshop on Compositional Theory and Technology for Real-Time Embedded Systems*, Barcelona, Spain, December 2008. IEEE Computer Society Press.

[13] J. Lehoczky, L. Sha, and Y. Ding. The rate monotonic scheduling algorithm: Exact characterization and average case behavior. In *Proceedings of the Real-Time Systems Symposium - 1989*, pages 166–171, Santa Monica, California, USA, Dec. 1989. IEEE Computer Society Press.

[14] G. Lipari and E. Bini. Resource partitioning among real-time applications. In *Proceedings of the EuroMicro Conference on Real-time Systems*, pages 151–160, Porto, Portugal, 2003. IEEE Computer Society.

[15] A. Mok. Task management techniques for enforcing ED scheduling on a periodic task set. In *Proc. 5th IEEE Workshop on Real-Time Software and Operating Systems*, pages 42–46, Washington D.C., May 1988.

[16] A. K. Mok. *Fundamental Design Problems of Distributed Systems for The Hard-Real-Time Environment*. PhD thesis, Laboratory for Computer Science, Massachusetts Institute of Technology, 1983. Available as Technical Report No. MIT/LCS/TR-297.

[17] R. Rajkumar, K. Juvva, A. Molano, and S. Oikawa. Resource kernels: a resource-centric approach to real-time and multimedia systems. In *Readings in multimedia computing and networking*, pages 476–490. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2001.

[18] I. Shin, A. Easwaran, and I. Lee. Hierarchical scheduling framework for virtual clustering of multiprocessors. In *Proceedings of the EuroMicro Conference on Real-Time Systems*, Prague, Czech Republic, July 2008. IEEE Computer Society Press.

[19] I. Shin and I. Lee. Periodic resource model for compositional real-time guarantees. In *Proceedings of the IEEE Real-Time Systems Symposium*, pages 2–13. IEEE Computer Society, 2003.

[20] I. Shin and I. Lee. Compositional real-time scheduling framework. In *Proceedings of the IEEE Real-Time Systems Symposium*, pages 57–67. IEEE Computer Society, 2004.

[21] I. Shin and I. Lee. Compositional real-time scheduling framework with periodic model. *ACM Transactions on Embedded Computing Systems*, 7(3), April 2008.

[22] V. V. Vazirani. *Approximation Algorithms*. Springer-Verlag, Berlin-Heidelberg-New York-Barcelona-Hong Kong-London-Milan-Paris-Singapur-Tokyo, 2001.

[23] E. Wandeler and L. Thiele. Real-time interfaces for interface-based design of real-time systems with fixed priority scheduling. In *EMSOFT '05: Proceedings of the 5th ACM international conference on Embedded software*, pages 80–89, New York, NY, USA, 2005. ACM.