Network Design: Problem Modeling

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Outline

- Basic Design Problems
- Routing Restriction
- Dimensioning for Modular Link Capacity
- Additional considerations
  - Nonlinear Link Dimensioning, Cost and Delay functions
  - Budget Constraint
  - Incremental NDP
  - Extensions
Outline

- Basic Design Problems
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  - Extensions
NDP Modeling

- Design for normal (nominal) operating state
  - average demand volumes, no variation
  - resource fully available, no failure.

- Two time scales
  - Uncapacitated design
    - for a given demand, *how much resource needed and how to distribute*
    - medium/long term planning
  - Capacitated design
    - given demand, resource, how to *allocate flows to paths* to optimize a network goal
    - short/medium term design
Dimensioning/Simple Design Problem (D/SDP): Link-Path Formulation

Simple Design Problem

- indices
  \[ d = 1, 2, \ldots, D \] demands
  \[ p = 1, 2, \ldots, P_d \] candidate paths for flows realizing demand \( d \)
  \[ e = 1, 2, \ldots, E \] links

- constants
  \[ \delta_{epd} \] = 1, if link \( e \) belongs to path \( p \) realizing demand \( d \); 0, otherwise
  \[ h_d \] volume of demand \( d \)
  \[ \xi_e \] unit (marginal) cost of link \( e \)

- variables
  \[ x_{dp} \] flow allocated to path \( p \) of demand \( d \) (continuous non-negative)
  \[ y_e \] capacity of link \( e \) (continuous non-negative)

- objective
  minimize \( F = \sum_{e} \xi_{e} y_{e} \) (bandwidth cost)

- constraints
  \[ \sum_{p} x_{dp} = h_{d}, \quad d = 1, 2, \ldots, D \] (demand constraints)
  \[ \sum_{d} \sum_{p} \delta_{epd} x_{dp} \leq y_{e}, \quad e = 1, 2, \ldots, E \] (capacity constraints).

Shortest path allocation rule: allocate all volume to cheapest path.
Dimensioning/Simple Design Problem: Node-Link Formulation I: link flow

• constants

- \(a_{ev}\) = 1 if link \(e\) originates at node \(v\), 0 otherwise
- \(b_{ev}\) = 1 if link \(e\) terminates in node \(v\), 0 otherwise
- \(s_d\) source node of demand \(d\)
- \(t_d\) sink node of demand \(d\)
- \(h_d\) volume of demand \(d\)
- \(\xi_e\) unit cost of link \(e\)

• variables

- \(x_{ed}\) flow realizing demand \(d\) allocated to link \(e\) (continuous non-negative)
- \(y_e\) capacity of link \(e\) (continuous non-negative)

• objective

minimize \(F = \sum_e \xi_e y_e\)

• constraints

\[\sum_e a_{ev} x_{ed} - \sum_e b_{ev} x_{ed} = \begin{cases} h_d, & \text{if } v = s_d \\ 0, & \text{if } v \neq s_d, t_d, \quad v = 1, 2, \ldots, V; d = 1, 2, \ldots, D \\ -h_d, & \text{if } v = t_d \end{cases}\]

\[\sum_d x_{ed} \leq y_e, \quad e = 1, 2, \ldots, E.\]
Dimensioning/Simple Design Problem: Node-Link Formulation II: node flow (fewer flow variables than link flow)

- **constants**
  - \( a_{ev} \): 1 if link \( e \) originates at node \( v \), 0 otherwise
  - \( b_{ev} \): 1 if link \( e \) terminates in node \( v \), 0 otherwise
  - \( h_{vv'} \): volume of demand \( d \) originating at node \( v \) and terminating at node \( v' \)
  - \( H_v \): \( \sum_{v' \neq v} h_{vv'} \) - total demand volume originating in node \( v \)
  - \( \xi_e \): unit cost of link \( e \)

- **variables**
  - \( x_{ev} \): flow realizing all demands originating at node \( v \) on link \( e \)
  - \( y_e \): capacity of link \( e \)

- **objective**
  - minimize \( F = \sum_e \xi_e y_e \)

- **constraints**
  - \( \sum_e a_{ev} x_{ev} = H_v, \quad v = 1, 2, ..., V \) (focus on \( v \))
  - \( \sum_e b_{ev} x_{ev} - \sum_e a_{ev'} x_{ev} = h_{vv'}, \quad v, v' = 1, 2, ..., V, \quad v \neq v' \) (focus on \( v' \))
  - \( \sum_v x_{ev} \leq y_e, \quad e = 1, 2, ..., E \).
Model Comparison

- Complexity

  - Table 4.1 Model Comparison

<table>
<thead>
<tr>
<th></th>
<th>Number of Variables</th>
<th>Number of Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Link-path formulation</td>
<td>$P \times V'(V' - 1) + \frac{1}{2} k \times V = O(V^2)$</td>
<td>$P \times V'(V' - 1) + \frac{1}{2} k \times V = O(V^2)$</td>
</tr>
<tr>
<td>Node-link formulation</td>
<td>$\frac{1}{2} k \times V \times V'(V' - 1) = O(V^3)$</td>
<td>$V \times V'(V' - 1) + \frac{1}{2} k \times V = O(V^2)$</td>
</tr>
<tr>
<td>Modified node-link</td>
<td>$\frac{1}{2} k \times V \times (V' + 1) = O(V^2)$</td>
<td>$V'(V + 1) + \frac{1}{2} k \times V = O(V^2)$</td>
</tr>
</tbody>
</table>

- Flexibility

  - Link-path formulation (PF): pre-compute path
  - Node-link formulation (LF): implicitly all possible paths
  - Path eliminating
    - PF: exclude in path pre-processing, set path flow to zero
    - LF: manipulate link flow to control path flow, e.g., prevent certain link flow

![Network with Separate End and Transit Nodes](image)
Capacitated Problem

 Allocation problem (A): given link capacities, whether demands are realizable?

Pure Allocation Problem

- **indices**
  - \( d = 1, 2, ..., D \) demands
  - \( p = 1, 2, ..., P_d \) candidate paths for flows realizing demand \( d \)
  - \( e = 1, 2, ..., E \) links

- **constants**
  - \( \delta_{edp} \) = 1 if link \( e \) belongs to path \( p \) realizing demand \( d \); = 0 otherwise
  - \( h_d \) volume of demand \( d \)
  - \( c_e \) capacity of link \( e \)

- **variables**
  - \( x_{dp} \) flow allocated to path \( p \) of demand \( d \) (continuous non-negative)

- **constraints**
  - \( \sum_p x_{dp} = h_d, \quad d = 1, 2, ..., D \)
  - \( \sum_d \sum_p \delta_{edp} x_{dp} \leq c_e, \quad e = 1, 2, ..., E. \)
Modified Link-Path Formulation

- how much additional bandwidth needed on each link to accommodate current demand?

PAP – Modified Link-Path Formulation

- **indices**
  
  \[ d = 1, 2, \ldots, D \] \hspace{1cm} \text{demands}

  \[ p = 1, 2, \ldots, P_d \] \hspace{1cm} \text{candidate paths for flows realizing demand } d

  \[ e = 1, 2, \ldots, E \] \hspace{1cm} \text{links}

- **constants**

  \[ \delta_{edp} = 1 \text{ if link } e \text{ belongs to path } p \text{ realizing demand } d; = 0 \text{ otherwise} \]

  \[ h_d \] \hspace{1cm} \text{volume of demand } d

  \[ c_e \] \hspace{1cm} \text{capacity of link } e

- **variables**

  \[ x_{dp} \] \hspace{1cm} \text{flow allocated to path } p \text{ of demand } d

  \[ z \] \hspace{1cm} \text{auxiliary continuous variable (of unrestricted sign)}

- **objective**

  minimize \( z \)

- **constraints**

  \[ \sum_p x_{dp} = h_d, \quad d = 1, 2, \ldots, D \]

  \[ \sum_d \sum_p \delta_{edp} x_{dp} \leq z + c_e, \]
How many paths needed?

- Proposition: If there is a feasible allocation, then there exists an allocation with at most \(D+E\) non-zero flows
  - \(D\) flows if all links are unsaturated

- Assign the entire demand volume of each demand to one of its shortest paths (\#hops); if all links are saturated and at least one is overloaded in the resulting solution, then there is no feasible allocation. (exercise)
Proposition 4.1: If the problem is feasible, then \textit{at most} $D+E$ flow variables are required to be nonzero at optimality.

- See book for proof (Page 113)
- Basic idea:
  - use “auxiliary, slack variable” to transform the link capacity constraint in “inequality” form to “equality form”;
  - # of non-zero (basic) variables in any basic feasible solution is at most equal to the number of equations.

Illustration: $N=50$, $E=200$ example

- Here $D = \frac{1225}{2} = \frac{N(N-1)}{2}$
- $D+E = 1425$
- Each pair must have one positive path flow
  - => remains 200 to be positive, thus only about 200 demand pairs would have more than one positive path flow at optimality
Mixed Capacitated/Uncapacitated Problem

- with *upper bounds* on link capacities

**Bounded Link Capacities**

- **indices**
  - $d = 1, 2, \ldots, D$ demands
  - $p = 1, 2, \ldots, P_d$ candidate paths for flows realizing demand $d$
  - $e = 1, 2, \ldots, E$ links

- **constants**
  - $\delta_{e dp} = 1$ if link $e$ belongs to path $p$ realizing demand $d$; 0, otherwise
  - $h_d$ volume of demand $d$
  - $c_e$ upper bound on the capacity of link
  - $\xi_e$ unit cost of link $e$

- **variables**
  - $x_{dp}$ flow allocated to path $p$ of demand (continuous non-negative) $d$
  - $y_e$ capacity of link $e$ (continuous non-negative)

- **objective**
  - minimize $F = \sum_e \xi_e y_e$

- **constraints**
  - $\sum_p x_{dp} = h_d$, $d = 1, 2, \ldots, D$
  - $\sum_d \sum_p \delta_{e dp} x_{dp} \leq y_e$, $e = 1, 2, \ldots, E$
  - $y_e \leq c_e$, $e = 1, 2, \ldots, E$. 
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Introducing Routing Restriction

- enforce the resulting routes w./w.o. certain properties
  - path diversity vs. limited split
  - equal splitting vs. arbitrary splitting
  - modular flows vs. unmodular/real flows

- extend the basic formulation by introducing additional routing constraints
Path Diversity

“never put all eggs in one basket” (?)

Generalized Diversity

• indices
  \[ d = 1, 2, \ldots, D \] demands
  \[ p = 1, 2, \ldots, P_d \] candidate paths for flows realizing demand \( d \)
  \[ e = 1, 2, \ldots, E \] links

• constants
  \[ \delta_{edp} \] = 1 if link \( e \) belongs to path \( p \) realizing demand \( d \); 0, otherwise
  \[ h_d \] volume of demand \( d \)
  \[ n_d \] diversity factor for demand \( d \)
  \[ c_e \] capacity of link \( e \)

• variables
  \[ x_{dp} \] flow allocated to path \( p \) of demand \( d \)

• constraints
  \[ \sum_p x_{dp} = h_d, \quad d = 1, 2, \ldots, D \]
  \[ \sum_d \sum_p \delta_{edp} x_{dp} \leq c_e, \quad e = 1, 2, \ldots, E \]
  \[ \sum_p \delta_{edp} x_{dp} \leq h_d / n_d, \quad e = 1, 2, \ldots, E \quad d = 1, 2, \ldots, D. \]
Lower Bounds on Non-Zero Flows

- the flow volume on a path, if any, should be lower bounded (?)
  - implicitly limit number of paths

### Lower-Bounded Flows

- **indices**
  - \( d = 1, 2, ..., D \) demands
  - \( p = 1, 2, ..., P_d \) candidate paths for flows realizing demand \( d \)
  - \( e = 1, 2, ..., E \) links

- **constants**
  - \( \delta_{edp} \) = 1 if link \( e \) belongs to path \( p \) realizing demand \( d \); 0, otherwise
  - \( h_d \) volume of demand \( d \)
  - \( b_d \) lower bound on non-zero flows of demand \( d \)
  - \( c_e \) capacity of link \( e \)

- **variables**
  - \( x_{dp} \) continuous flow variable allocated to path \( p \) of demand \( d \)
  - \( u_{dp} \) binary variable corresponding to \( x_{dp} \)

- **constraints**
  - \( \sum_p x_{dp} = h_d; \quad d = 1, 2, ..., D \)
  - \( x_{dp} \leq h_d u_{dp}; \quad d = 1, 2, ..., D \quad p = 1, 2, ..., P_d \)
  - \( b_d u_{dp} \leq x_{dp}; \quad d = 1, 2, ..., D \quad p = 1, 2, ..., P_d \)
  - \( \sum_d \sum_p \delta_{edp} x_{dp} \leq c_e; \quad e = 1, 2, ..., E \).
Limited Demand Split

- only split among $k$ paths; $k=1 \Rightarrow \text{single path allocation (?)}$

**Single-Path Allocation**

- **indices**
  - $d = 1, 2, \ldots, D$ demands
  - $p = 1, 2, \ldots, P_d$ candidate paths for flows realizing demand $d$
  - $e = 1, 2, \ldots, E$ links

- **constants**
  - $\delta_{edp} = 1$ if link $e$ belongs to path $p$ realizing demand $d$; 0, otherwise
  - $h_d$ volume of demand $d$
  - $c_e$ capacity of link $e$

- **variables**
  - $x_{dp}$ flow allocated to path $p$ of demand $d$
  - $u_{dp}$ binary variable associated with flow $x_{dp}$

- **constraints**
  - $x_{dp} = h_d u_{dp}, \quad d = 1, 2, \ldots, D \quad p = 1, \ldots, P_d$
  - $\sum_p u_{dp} = 1, \quad d = 1, 2, \ldots, D$
  - $\sum_d \sum_p \delta_{edp} x_{dp} \leq c_e, \quad e = 1, 2, \ldots, E.$

- The above “single path allocation” problem is NP-complete
  - Reduction from “integral flow problem w/ homogeneous unit demands”, which in turn is proved by reduction from “two demand, integral flow problem”
Node-Link Formulation

- **Single Path allocation**

- **constants**
  - \(a_{ev}\) = 1 if node \(v\) is the originating node of link \(e\); 0, otherwise
  - \(b_{ev}\) = 1 if node \(v\) is the terminating node of link \(e\); 0, otherwise
  - \(s_d\) source node of demand \(d\)
  - \(t_d\) sink node of demand \(d\)
  - \(h_d\) volume of demand \(d\)
  - \(c_e\) capacity of link \(e\)

- **variables**
  - \(u_{de}\) binary variable corresponding to flow of demand \(d\) allocated to link \(e\)

- **constraints**
  
  \[
  \sum_d h_d u_{de} \leq c_e, \quad e = 1, 2, \ldots, E
  \]
  \[
  \sum_e a_{ev} u_{de} - \sum_e b_{ev} u_{de} = \begin{cases} 
  1, & \text{if } v = s_d \\
  0, & \text{if } v \neq s_d, t_d, \\
  -1, & \text{if } v = t_d.
  \end{cases} \quad v = 1, 2, \ldots, V; d = 1, 2, \ldots, D
  \]
Node-Link Formulation

- equally split among \( k \) link-disjoint paths

**Equal Split Among \( k \) Link-Disjoint Paths**

- **indices**
  
  \[
  \begin{align*}
  d &= 1, 2, \ldots, D & \text{demands} \\
  e &= 1, 2, \ldots, E & \text{links} \\
  v &= 1, 2, \ldots, V & \text{nodes}
  \end{align*}
  \]

- **constants**

  \[
  \begin{align*}
  a_{ev} &= 1 \text{ if node } v \text{ is the originating node of link } e; 0, \text{ otherwise} \\
  b_{ev} &= 1 \text{ if node } v \text{ is the terminating node of link } e; 0, \text{ otherwise} \\
  s_d &= \text{source node of demand } d \\
  t_d &= \text{sink node of demand } d \\
  h_d &= \text{volume of demand } d \\
  c_e &= \text{capacity of link } e \\
  k_d &= \text{predetermined number of paths for demand } d
  \end{align*}
  \]

- **variables**

  \[
  u_{de} = \text{binary variable corresponding to flow of demand } d \text{ allocated to link } e
  \]

- **constraints**

  \[
  \begin{align*}
  \sum_d u_{de} h_d / k_d & \leq c_e, & e = 1, 2, \ldots, E \\
  \sum_e a_{ev} u_{de} - \sum_e b_{ev} u_{de} & = \begin{cases} 
  k_d, & \text{if } v = s_d \\
  0, & \text{if } v \neq s_d, t_d \\
  -k_d, & \text{if } v = t_d.
  \end{cases} & v = 1, 2, \ldots, V; d = 1, 2, \ldots, D
  \end{align*}
  \]
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**Integral Flows**

- allocate demand volumes in demand modules

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**Modular Flow Allocation**

- **indices**
  - \( d = 1, 2, ..., D \) demands
  - \( p = 1, 2, ..., P_d \) candidate paths for flows realizing demand \( d \)
  - \( e = 1, 2, ..., E \) links

- **constants**
  - \( \delta_{edp} \) = 1 if link \( e \) belongs to path \( p \) realizing demand \( d \); 0 otherwise
  - \( L_d \) demand module for demand \( d \)
  - \( H_d \) volume of demand \( d \) expressed as the number of demand modules
  - \( h_d \) demand volume \( (h_d = L_d H_d) \)
  - \( c_e \) capacity of link \( e \)

- **variables**
  - \( u_{dp} \) non-negative integral variable associated with the flow on path \( p \) of demand \( d \)

- **constraints**
  - \( \sum_p u_{dp} = H_d, \quad d = 1, 2, ..., D \)
  - \( \sum_d L_d \sum_p \delta_{edp} u_{dp} \leq c_e, \quad e = 1, 2, ..., E. \)
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Nonlinear Link Cost

- **Linear Link Cost**
  - link capacity = link rate
  - linear cost: $/bps

- **Nonlinear Link Cost**
  - modular link capacities
  - different link modules

**Figure 4.2** Cost of a Modular Link Versus Link Load

**Figure 4.3** Cost of a Link with Multiple Modules
Dimensioning with Modular Links

**Modular Links**

- **indices**
  - \( d = 1, 2, \ldots, D \) demands
  - \( p = 1, 2, \ldots, P_d \) candidate paths for flow realizing demand \( d \)
  - \( e = 1, 2, \ldots, E \) links

- **constants**
  - \( \delta_{edp} \) = 1, if link \( e \) belongs to path \( p \) realizing demand \( d \), 0 otherwise
  - \( h_d \) volume of demand \( d \)
  - \( \xi_e \) cost of one capacity module on link \( e \)
  - \( M \) size of the link capacity module

- **variables**
  - \( x_{dp} \) flow allocated to path \( p \) of demand \( d \) (continuous non-negative)
  - \( y_e \) capacity of link \( e \) expressed in the number of modules (non-negative integer)

- **objective**
  - minimize \( F = \sum_e \xi_e y_e \)

- **constraints**
  - \( \sum_p x_{dp} = h_d, \quad d = 1, 2, \ldots, D \)
  - \( \sum_d \sum_p \delta_{edp} x_{dp} \leq M y_e, \quad e = 1, 2, \ldots, E. \)

**NP-complete**
Dimensioning with Multiple Modules (?)

**Link With Multiple Modular Sizes**

- **indices**
  - \( d = 1, 2, \ldots, D \) demands
  - \( p = 1, 2, \ldots, P_d \) candidate paths for flow realizing demand \( d \)
  - \( e = 1, 2, \ldots, E \) links
  - \( k = 1, 2, \ldots, K \) types of modules

- **constants**
  - \( \delta_{edp} \) = 1, if link \( e \) belongs to path \( p \) realizing demand \( d \), 0 otherwise
  - \( h_d \) volume of demand \( d \)
  - \( \xi_{ek} \) cost of one capacity module of type \( k \) on link \( e \)
  - \( M_k \) size of the link capacity module of type \( k \)

- **variables**
  - \( x_{dp} \) flow allocated to path \( p \) of demand \( d \)
  - \( y_{ek} \) number of modules of type \( k \) on link \( e \) (non-negative integer)

- **objective**
  - minimize \( F = \sum_e \sum_k \xi_{ek} y_{ek} \)

- **constraints**
  - \( \sum_p x_{dp} = h_d, \quad d = 1, 2, \ldots, D \)
  - \( \sum_d \sum_p \delta_{edp} x_{dp} \leq \sum_k M_k y_{ek}, \quad e = 1, 2, \ldots, E. \)

NP-complete?
Convex Cost Functions

- Convex Function
  - \((1 - \alpha)f(x) + \alpha f(y) \geq f((1 - \alpha)x + \alpha y), \forall \alpha \in [0, 1]\)
  - \(f(y) \geq f(x) + \nabla f(x)(y - x)\)
  - non-negative second order derivative
  - local minimum \(\Rightarrow\) global minimum

- good approx. for link delay
  - \(F_e(y_e) = \frac{1}{c_e - y_e}, \quad 0 \leq y_e < c_e\)

- split demand if possible
  - Because \(f(z_1)/z_1 \leq f(z_2)/z_2\) for \(z_1 < z_2\)

- how to split?
Minimal Delay Routing

- link delay, network delay, avg. user delay

**Convex Cost Function**

- **indices**
  
  \[ \begin{align*}
  d &= 1, 2, \ldots, D \quad \text{demands} \\
  p &= 1, 2, \ldots, P_d \quad \text{candidate paths for flows realizing demand } d \\
  e &= 1, 2, \ldots, E \quad \text{links}
  \end{align*} \]

- **constants**
  
  \[ \begin{align*}
  \delta_{epd} &= 1 \text{ if link } e \text{ belongs to path } p \text{ realizing demand } d; 0, \text{ otherwise} \\
  h_d &= \text{volume of demand } d \\
  F_e(\cdot) &= \text{convex cost function of link } e \\
  c_e &= \text{capacity of link } e
  \end{align*} \]

- **variables**
  
  \[ \begin{align*}
  x_{dp} &= \text{flow allocated to path } p \text{ of demand } d \text{ (continuous non-negative)} \\
  y_e &= \text{load of link } e \text{ (continuous non-negative)}
  \end{align*} \]

- **objective**
  
  minimize \( F = \sum_e F_e(y_e) \)

- **constraints**
  
  \[ \begin{align*}
  \sum_p x_{dp} &= h_d, \quad d = 1, 2, \ldots, D \\
  \sum_d \sum_p \delta_{epd} x_{dp} &= y_e, \quad e = 1, 2, \ldots, E \\
  y_e &\leq c_e, \quad e = 1, 2, \ldots, E.
  \end{align*} \]
Piecewise Linear Approximation of Convex Function

\[ f(z) = \begin{cases} 
0 & 0 \leq z \leq 1 \\
(z - 1)^2 & z > 1.
\end{cases} \]

\[ \hat{f}(z) = \begin{cases} 
0 & 0 \leq z \leq 1 \\
z - 1 & 1 \leq z < 2 \\
3(z - 2) + 1 = 3z - 5 & 2 \leq z < 3 \\
10(z - 3) + 4 = 10z - 26 & z \geq 3.
\end{cases} \]
Piecewise Linear Approximation of Convex Function

\[ f(y) = \max_{k=1,2,\ldots,K} a_k y + b_k \]

\[ f(y) = \begin{cases} 
\text{minimize} & r \\
\text{subject to} & r \geq a_k y + b_k, \quad k = 1, 2, \ldots, K.
\end{cases} \]
Convex Penalty Function – Piecewise Linear Approximation

- **indices**
  - \( d = 1, 2, \ldots, D \) demands
  - \( p = 1, 2, \ldots, P_d \) candidate paths for flows realizing demand \( d \)
  - \( e = 1, 2, \ldots, E \) links
  - \( k = 1, 2, \ldots, K_e \) consecutive pieces of the linear approximation of \( F_e(\cdot) \)

- **constants**
  - \( \delta_{edp} \) = 1 if link \( e \) belongs to path \( p \) realizing demand \( d \); 0, otherwise
  - \( h_d \) volume of demand \( d \)
  - \( c_e \) capacity of link \( e \)
  - \( F_e(\cdot) \) convex penalty function of link \( e \)
  - \( a_{ek}, b_{ek} \) coefficients of the linear pieces of the piecewise linear approximation of \( F_e(\cdot) \)

- **variables**
  - \( x_{dp} \) flow allocated to path \( p \) of demand \( d \)
  - \( y_e \) load of link \( e \)
  - \( r_e \) continuous variable approximating \( F_e(y_e) \)

- **objective**
  - minimize \( F = \sum_e r_e \)

- **constraints**
  - \( \sum_p x_{dp} = h_d, \quad d = 1, 2, \ldots, D \)
  - \( \sum_d \sum_p \delta_{edp} x_{dp} = y_e, \quad e = 1, 2, \ldots, E \)
  - \( r_e > a_{ek} y_e + b_{ek}, \quad e = 1, 2, \ldots, E \quad k = 1, 2, \ldots, K_e \)
Concave Link Dimensioning Functions

- **Concave Function**
  - \( \alpha f(z_1) + (1-\alpha) f(z_2) \leq f(\alpha z_1 + (1-\alpha) z_2), \forall \alpha \in [0, 1] \)
  - non-positive second derivative, unique maximum
  - Erlang B-Loss Formula (extend to real domain)
    \[
    b_e = B(A, y_e), \quad y_e = B^{-1}(A, b_e)
    \]

- **Implications**
  - merge resource if possible
    - Because \( f(z_1)/z_1 \geq f(z_2)/z_2 \) for \( z_1 < z_2 \)
Figure 4.7  Piecewise Linear Approximation of a Concave Function
Concave Link Dimensioning

Concave Dimensioning Functions

- **indices**
  
  \[ d = 1, 2, \ldots, D \quad \text{demands} \]
  
  \[ p = 1, 2, \ldots, P_d \quad \text{candidate paths for flows realizing demand } d \]
  
  \[ e = 1, 2, \ldots, E \quad \text{links} \]

- **constants**
  
  \[ \delta_{edp} = 1 \text{ if link } e \text{ belongs to path } p \text{ realizing demand } d; \ 0, \text{ otherwise} \]
  
  \[ h_d \quad \text{volume of demand } d \]
  
  \[ \xi_e \quad \text{unit cost of link } e \]
  
  \[ F_e(\cdot) \quad \text{non-decreasing concave dimensioning function of link } e \]

- **variables**
  
  \[ x_{dp} \quad \text{flow allocated to path } p \text{ of demand } d \]
  
  \[ y_e \quad \text{load of link } e \]

- **objective**
  
  minimize \[ F = \sum_e \xi_e F_e(y_e) \]

- **constraints**
  
  \[ \sum_p x_{dp} = h_d, \quad d = 1, 2, \ldots, D \]
  
  \[ \sum_d \sum_p \delta_{edp} x_{dp} = y_e, \quad e = 1, 2, \ldots, E. \]
Outline

- Basic Design Problems
- Routing Restriction
- Dimensioning for Modular Link Capacity
- Additional considerations
  - Nonlinear Link Dimensioning, Cost and Delay functions
  - Budget Constraint
  - Incremental NDP
  - Extensions
Budget Constraint

- given budget constraint $B$, maximize the realized ratio $r$ for all demands.

- objective
  maximize $r$

- constraints
  \[ \sum_p x_{dp} \geq r h_d, \quad d = 1, 2, ..., D \]
  \[ \sum_d \sum_p \delta_{edp} x_{dp} \leq y_e, \quad e = 1, 2, ..., E \]
  \[ \sum_e \xi_e y_e \leq B. \]
Outline

- Basic Design Problems
- Routing Restriction
- Dimensioning for Modular Link Capacity
- Additional considerations
  - Nonlinear Link Dimensioning, Cost and Delay functions
  - Budget Constraint
  - Incremental NDP
  - Extensions
Incremental NDPs

- design from scratch v.s. improve existing network; sub-optimal solution

Simple Extension Problem

- indices
  
  \[ d = 1, 2, \ldots, D \]  
  demands

  \[ p = 1, 2, \ldots, P_d \]  
  candidate paths for flow realizing demand \( d \)

  \[ e = 1, 2, \ldots, E \]  
  links

- constants
  
  \( \delta_{edp} = 1, \) if link \( e \) belongs to path \( p \) realizing demand \( d \), \( 0 \) otherwise

  \( h_d \)  
  volume of demand \( d \)

  \( \xi_e \)  
  unit cost of link \( e \)

  \( c_e \)  
  existing capacity of link \( e \)

- variables
  
  \( x_{dp} \)  
  flow variable allocated to path \( p \) of demand \( d \) (continuous non-negative)

  \( y_e \)  
  extra capacity of link \( e \) on top of \( c_e \) (continuous non-negative)

- objective
  
  minimize \( F = \sum_e \xi_e y_e \)

- constraints
  
  \[ \sum_p x_{dp} = h_d, \quad d = 1, 2, \ldots, D \]

  \[ \sum_d \sum_p \delta_{edp} x_{dp} \leq y_e + c_e, \quad e = 1, 2, \ldots, E. \]
Outline

- Basic Design Problems
- Routing Restriction
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- Additional considerations
  - Nonlinear Link Dimensioning, Cost and Delay functions
  - Budget Constraint
  - Incremental NDP
  - Extensions
Extensions: nodes

- constraints on nodes
  - node cost: input/output ports, link termination, switching fabric, installation, ...
  - reliability: node disjoint

- virtual link (graph)
  - two copies for a node: receiving/sending
  - directed link from receiving copy to sending copy

- incorporating node constraints
  - node cost represented by link cost on its virtual link
  - node-disjoint in real graph <=> link-disjoint in virtual graph
Extensions: nodes (contd.)

- link-path formulation

\[ \delta_{e(v)dp} = \begin{cases} 1, & \text{if node } v \text{ belongs to path } p \text{ realizing demand } d \\ 0, & \text{otherwise} \end{cases} \]

- load on a node: \[ y_v = \sum_{d} \sum_{p} (\delta_{e(v)dp} x_{dp}) \]

- reliability against node failures: no node carries more than certain share for a demand
  - link-path formulation \[ \sum_{p} (\delta_{e(v)dp} x_{dp}) \leq h_d/n_d, \quad v \neq s_d, v \neq t_d \]
  - node-link formulation \[ x_{e(v)d} \leq h_d/n_d, \quad v \neq s_d, v \neq t_d \]
Summary

- Basic Design Problems
  - Uncapacitated & capacitated

- Routing Restriction

- Dimensioning for Modular Link Capacity

- Additional considerations
  - Nonlinear Link Dimensioning, Cost and Delay functions
  - Budget Constraint
  - Incremental NDP
  - Extensions
Exercise #1:
- Exercises 4.6, 4.7, 4.23

TinyExam #1