Time Reversibility and Burke’s Theorem

Hongwei Zhang
http://www.cs.wayne.edu/~hzhang

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Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke’s Theorem
- Queues in Tandem
Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke’s Theorem
- Queues in Tandem
Time-Reversed Markov Chains

- \( \{X_n: n=0,1,...\} \) irreducible aperiodic Markov chain with transition probabilities \( P_{ij} \)
  \[
  \sum_{j=0}^{\infty} P_{ij} = 1, \quad i = 0,1,...
  \]
- Unique stationary distribution \( (\pi_j > 0) \) if and only if GBE holds, i.e.,
  \[
  \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0,1,...
  \]
- Process in steady state:
  \[
  \Pr\{X_n = j\} = \pi_j = \lim_{n \to \infty} \Pr\{X_n = j \mid X_0 = i\}
  \]
  - Starts at \( n=-\infty \), that is \( \{X_n: n = ...,-1,0,1,...\} \), or
  - Choose initial state according to the stationary distribution

- How does \( \{X_n\} \) look “reversed” in time?
Time-Reversed Markov Chains

- Define $Y_n = X_{\tau-n}$, for arbitrary $\tau > 0$
  
  $\Rightarrow \{Y_n\}$ is the reversed process.

- Proposition 1:
  - $\{Y_n\}$ is a Markov chain with transition probabilities:
    
    $$P^*_ij = \frac{\pi_j \cdot P_{ji}}{\pi_i}, \quad i, j = 0, 1, ...$$

  - $\{Y_n\}$ has the same stationary distribution $\pi_j$ with the forward chain $\{X_n\}$
    
    The reversed chain corresponds to the same process, looked at in the reversed-time direction
Time-Reversed Markov Chains

Proof of Proposition 1:

\[ P_{ij}^* = P\{ Y_m = j \mid Y_{m-1} = i, Y_{m-2} = i_2, \ldots, Y_{m-k} = i_k \} \]

\[ = P\{ X_{\tau-m} = j \mid X_{\tau-m+1} = i, X_{\tau-m+2} = i_2, \ldots, X_{\tau-m+k} = i_k \} \]

\[ = P\{ X_n = j \mid X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k \} \]

\[ = \frac{P\{ X_n = j, X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k \}}{P\{ X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k \}} \]

\[ = \frac{P\{ X_{n+2} = i_2, \ldots, X_{n+k} = i_k \mid X_n = j, X_{n+1} = i \}}{P\{ X_{n+1} = i \}} \]

\[ = \frac{P\{ X_n = j, X_{n+1} = i \}}{P\{ X_{n+1} = i \}} = P\{ X_n = j \mid X_{n+1} = i \} = P\{ Y_m = j \mid Y_{m-1} = i \} \]

\[ = \frac{P\{ X_{n+1} = i \mid X_n = j \}}{P\{ X_{n+1} = i \}} = \frac{P_{ji}}{\pi_i} \]

\[ \sum_{i=0}^{\infty} \pi_i P_{ij}^* = \sum_{i=0}^{\infty} \pi_i \frac{\pi_j P_{ji}}{\pi_i} = \pi_j \sum_{i=0}^{\infty} P_{ji} = \pi_j \]
Outline

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke’s Theorem
- Queues in Tandem
Reversibility

- Stochastic process \{X(t)\} is called *reversible* if \((X(t_1), X(t_2), \ldots, X(t_n))\) and \((X(\tau-t_1), X(\tau-t_2), \ldots, X(\tau-t_n))\) have the same probability distribution, for all \(\tau, t_1, \ldots, t_n\)

- Markov chain \(\{X_n\}\) is *reversible* if and only if the transition probabilities of forward and reversed chains are equal, i.e.,

\[ P_{ij} = P_{ji}^* \]

- Detailed Balance Equations \(\leftrightarrow\) Reversibility

\[ \pi_i P_{ij} = \pi_j P_{ji}, \quad i, j = 0, 1, \ldots \]
Reversibility – Discrete-Time Chains

- **Theorem 1:** If there exists a set of positive numbers \( \{\pi_j\} \), that sum up to 1 and satisfy:

\[
\pi_i P_{ij} = \pi_j P_{ji}, \quad i, j = 0,1,\ldots
\]

Then:
1. \( \{\pi_j\} \) is the unique stationary distribution
2. The Markov chain is reversible

- **Example:** Discrete-time birth-death processes are reversible, since they satisfy the DBE
Example: Birth-Death Process

- One-dimensional Markov chain with transitions only between neighboring states: \( P_{ij} = 0 \), if \(|i-j| > 1\)

- Detailed Balance Equations (DBE)
  \[
  \pi_i P_{i,n+1} = \pi_{i+1} P_{i+1,n} \quad n = 0, 1, ...
  \]

- Proof: GBE with \( S = \{0, 1, \ldots, n\} \) give:
  \[
  \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_j P_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_i P_{ij} \Rightarrow \pi_i P_{n,n+1} = \pi_{n+1} P_{n+1,n}
  \]
Theorem 2: Irreducible Markov chain with transition probabilities $P_{ij}$. If there exist:

- A set of transition probabilities $P_{ij}^*$, with $\Sigma_j P_{ij}^* = 1$, $i \geq 0$, and
- A set of positive numbers $\{\pi_j\}$, that sum up to 1, such that

$$\pi_i P_{ij}^* = \pi_j P_{ji}, \quad i, j \geq 0$$

Then:

- $P_{ij}^*$ are the transition probabilities of the reversed chain, and
- $\{\pi_j\}$ is the stationary distribution of the forward and the reversed chains

Remark: Used to find the stationary distribution, by guessing the transition probabilities of the reversed chain – even if the process is not reversible
Continuous-Time Markov Chains

- \{X(t): -\infty < t < \infty\} irreducible aperiodic Markov chain with transition rates \( q_{ij}, \ i \neq j \)
- Unique stationary distribution \((p_j > 0)\) if and only if:
  \[
p_j \sum_{i \neq j} q_{ji} = \sum_{i \neq j} p_i q_{ij}, \quad j = 0, 1, \ldots
  \]
- Process in steady state – e.g., started at \( t = -\infty \):
  \[
  \Pr\{X(t) = j\} = p_j = \lim_{t \to \infty} \Pr\{X(t) = j \mid X(0) = i\}
  \]
- If \( \{\pi_j\} \) is the stationary distribution of the embedded discrete-time chain:
  \[
p_j = \frac{\pi_j / \nu_j}{\sum_i \pi_i / \nu_i}, \quad \nu_j \equiv \sum_{i \neq j} q_{ji}, \quad j = 0, 1, \ldots
  \]
Reversal Continuous-Time Markov Chains

- Reversed chain \( \{Y(t)\} \), with \( Y(t) = X(\tau - t) \), for arbitrary \( \tau > 0 \)

- Proposition 2:
  1. \( \{Y(t)\} \) is a continuous-time Markov chain with transition rates:
     \[
     q_{ij}^* = \frac{p_j q_{ji}}{p_i}, \quad i, j = 0, 1, \ldots, i \neq j
     \]
  2. \( \{Y(t)\} \) has the same stationary distribution \( \{p_j\} \) with the forward chain

- Remark: The transition rate out of state \( i \) in the reversed chain is equal to the transition rate out of state \( i \) in the forward chain

\[
\sum_{j \neq i} q_{ij}^* = \frac{\sum_{j \neq i} p_j q_{ji}}{p_i} = \frac{p_i \sum_{j \neq i} q_{ij}}{p_i} = \sum_{j \neq i} q_{ij} = v_i, \quad i = 0, 1, \ldots
\]
Reversibility – Continuous-Time Chains

- Markov chain \( \{X(t)\} \) is *reversible* if and only if the transition rates of forward and reversed chains are equal \( q_{ij} = q_{ij}^* \), or equivalently

\[
p_i q_{ij} = p_j q_{ji}, \quad i, j = 0, 1, \ldots, i \neq j
\]

i.e., Detailed Balance Equations \( \leftrightarrow \) Reversibility

- **Theorem 3:** If there exists a set of positive numbers \( \{\rho_j\} \), that sum up to 1 and satisfy:

\[
p_i q_{ij} = p_j q_{ji}, \quad i, j = 0, 1, \ldots, i \neq j
\]

Then:

1. \( \{\rho_j\} \) is the unique stationary distribution
2. The Markov chain is reversible
Example: Birth-Death Process

Transitions only between neighboring states
\[ q_{i,i+1} = \lambda_i, \quad q_{i,i-1} = \mu_i, \quad q_{ij} = 0, \quad |i - j| > 1 \]

Detailed Balance Equations
\[ \lambda_n p_n = \mu_{n+1} p_{n+1}, \quad n = 0, 1, ... \]

Proof: GBE with \( S = \{0, 1, ..., n\} \) give:
\[ \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_j q_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_i q_{ij} \Rightarrow \lambda_n p_n = \mu_{n+1} p_{n+1} \]

- M/M/1, M/M/c, M/M/∞
Reversed Continuous-Time Markov Chains (Revisited)

- **Theorem 4:** Irreducible continuous-time Markov chain with transition rates $q_{ij}$. If there exist:
  - A set of transition rates $q_{ij}^*$, with $\sum_{j \neq i} q_{ij}^* = \sum_{j \neq i} q_{ij}$, $i \geq 0$, and
  - A set of positive numbers $\{p_j\}$, that sum up to 1, such that
    \[ p_i q_{ij}^* = p_j q_{ji}, \quad i, j \geq 0, i \neq j \]

Then:
- $q_{ij}^*$ are the transition rates of the reversed chain, and
- $\{p_j\}$ is the stationary distribution of the forward and the reversed chains

- **Remark:** Used to find the stationary distribution, by guessing the transition probabilities of the reversed chain – even if the process is not reversible
Reversibility: Trees

Theorem 5:
- Irreducible Markov chain, with transition rates that satisfy $q_{ij} > 0 \iff q_{ji} > 0$
- Form a graph for the chain, where states are the nodes, and for each $q_{ij} > 0$, there is a directed arc $i \rightarrow j$

Then, if graph is a tree – contains no loops – then Markov chain is reversible

Remarks:
- Sufficient condition for reversibility
- Generalization of one-dimensional birth-death process
Kolmogorov’s Criterion (Discrete Chain)

- Detailed balance equations determine whether a Markov chain is reversible or not, based on stationary distribution and transition probabilities.
- Should be able to derive a reversibility criterion based only on the transition probabilities!
- Theorem 6: A discrete-time Markov chain is reversible if and only if:

\[ P_{i_1i_2} P_{i_2i_3} \cdots P_{i_{n-1}i_n} P_{i_ni_1} = P_{i_1i_n} P_{i_ni_{n-1}} \cdots P_{i_{3i_2}} P_{i_2i_1} \]

for any finite sequence of states: \(i_1, i_2, \ldots, i_n\) and any \(n\).

- Intuition: Probability of traversing any loop \(i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_n \rightarrow i_1\) is equal to the probability of traversing the same loop in the reverse direction \(i_1 \rightarrow i_n \rightarrow \ldots \rightarrow i_2 \rightarrow i_1\).
Kolmogorov’s Criterion (Continuous Chain)

- Detailed balance equations determine whether a Markov chain is reversible or not, based on stationary distribution and \textit{transition rates}.

- Should be able to derive a reversibility criterion based only on the transition rates!

- \textbf{Theorem 7:} A continuous-time Markov chain is reversible \textit{if and only if}:

\[ q_{i_1i_2} q_{i_2i_3} \cdots q_{i_{n-1}i_n} q_{i_ni_1} = q_{i_1i_n} q_{i_ni_{n-1}} \cdots q_{i_3i_2} q_{i_2i_1} \]

for any finite sequence of states: \( i_1, i_2, \ldots, i_n \) and any \( n \)

- \textbf{Intuition:} Product of transition rates along any loop \( i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_n \rightarrow i_1 \) is equal to the product of transition rates along the same loop traversed in the reverse direction \( i_1 \rightarrow i_n \rightarrow \cdots \rightarrow i_2 \rightarrow i_1 \)
Kolmogorov’s Criterion (proof)

Proof of Theorem 6:

Necessary: If the chain is reversible the DBE hold

\[
\begin{align*}
\pi_1 P_{i_1i_2} &= \pi_2 P_{i_2i_1} \\
\pi_2 P_{i_2i_3} &= \pi_3 P_{i_3i_2} \\
&\quad \vdots \\
\pi_{n-1} P_{i_{n-1}i_n} &= \pi_n P_{i_ni_{n-1}} \\
\pi_n P_{i_ni_1} &= \pi_1 P_{i_1i_n}
\end{align*}
\]

\[\Rightarrow P_{i_1i_2} P_{i_2i_3} \cdots P_{i_{n-1}i_n} P_{i_ni_1} = P_{i_1i_n} P_{i_ni_{n-1}} \cdots P_{i_3i_2} P_{i_2i_1}\]

Sufficient: Fixing two states \(i_1=i\) and \(i_n=j\) and summing over all states \(i_2, \ldots, i_{n-1}\) we have

\[P_{i_1i_2} P_{i_2i_3} \cdots P_{i_{n-1}j} P_{ji} = P_{ij} P_{j,i_{n-1}} \cdots P_{i_3i_2} P_{i_2i_1} \Rightarrow P_{ij} P_{ji} = P_{ij} P_{ji}^{n-1}\]

Taking the limit \(n\to\infty\)

\[\lim_{n\to\infty} P_{ij}^{n-1} \cdot P_{ji} = P_{ij} \cdot \lim_{n\to\infty} P_{ji}^{n-1} \Rightarrow \pi_j P_{ji} = P_{ij} \pi_i\]
Example: M/M/2 Queue with Heterogeneous Servers

- M/M/2 queue. Servers A and B with service rates $\mu_A$ and $\mu_B$ respectively. When the system empty, arrivals go to A with probability $\alpha$ and to B with probability $1-\alpha$. Otherwise, the head of the queue takes the first free server.

- Need to keep track of which server is busy when there is 1 customer in the system. Denote the two possible states by: 1A and 1B.

- Reversibility: we only need to check the loop $0 \rightarrow 1A \rightarrow 2 \rightarrow 1B \rightarrow 0$:
  \[
  q_{0,1A}q_{1A,2}q_{2,1B}q_{1B,0} = \alpha \lambda \cdot \lambda \cdot \mu_A \cdot \mu_B \\
  q_{0,1B}q_{1B,2}q_{2,1A}q_{1A,0} = (1-\alpha) \lambda \cdot \lambda \cdot \mu_B \cdot \mu_A
  \]
  - Reversible if and only if $\alpha=1/2$.

What happens when $\mu_A=\mu_B$ and $\alpha \neq 1/2$?
Example: M/M/2 Queue with Heterogeneous Servers

\[ p_n = p_2 \left( \frac{\lambda}{\mu_A + \mu_B} \right)^{n-2}, \quad n = 2, 3, \ldots \]

\[ \begin{align*}
\lambda p_0 &= \mu_A p_{1A} + \mu_B p_{1B} \\
(\mu_A + \mu_B) p_2 &= \lambda (p_{1A} + p_{1B}) \\
(\mu_A + \lambda) p_{1A} &= \alpha \lambda p_0 + \mu_B p_2
\end{align*} \]

\[ p_{1A} = p_0 \left( \frac{\lambda + \alpha (\mu_A + \mu_B)}{\mu_A} \right) \]

\[ p_{1B} = p_0 \left( \frac{\lambda + (1 - \alpha)(\mu_A + \mu_B)}{\mu_B} \right) \]

\[ p_2 = p_0 \left( \frac{\lambda^2 + \lambda (1 - \alpha) \mu_A + \alpha \mu_B}{\mu_A \mu_B} \right) \]

\[ p_0 + p_{1A} + p_{1B} + \sum_{n=2}^{\infty} p_n = 1 \quad \Rightarrow \quad p_0 = \left[ 1 + \frac{\lambda}{\mu_A + \mu_B - \lambda} \right]^{-1} \left( \frac{\lambda^2 + \lambda (1 - \alpha) \mu_A + \alpha \mu_B}{\mu_A \mu_B} \right) \]
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- Time-Reversal of Markov Chains
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Multidimensional Markov Chains

Theorem 8:
- \( \{X_1(t)\}, \{X_2(t)\} \): independent Markov chains
- \( \{X(t)\} \): reversible
- \( \{X(t)\}, \text{ with } X(t) = (X_1(t), X_2(t)) \): vector-valued stochastic process
  - \( \{X(t)\} \) is a Markov chain
  - \( \{X(t)\} \) is reversible

Multidimensional Chains:
- Queueing system with two classes of customers, each having its own stochastic properties – track the number of customers from each class
- Study the “joint” evolution of two queueing systems – track the number of customers in each system
Example: Two Independent M/M/1 Queues

- Two independent M/M/1 queues. The arrival and service rates at queue $i$ are $\lambda_i$ and $\mu_i$ respectively. Assume $\rho_i = \frac{\lambda_i}{\mu_i} < 1$.

- $\{(N_1(t), N_2(t))\}$ is a Markov chain.

- Probability of $n_1$ customers at queue 1, and $n_2$ at queue 2, at steady-state

\[ p(n_1, n_2) = (1 - \rho_1)\rho_1^{n_1} \cdot (1 - \rho_2)\rho_2^{n_2} = p_1(n_1) \cdot p_2(n_2) \]

- “Product-form” distribution

- Generalizes for any number $K$ of independent queues, M/M/1, M/M/c, or M/M/$\infty$. If $p_i(n_i)$ is the stationary distribution of queue $i$:

\[ p(n_1, n_2, \ldots, n_K) = p_1(n_1) p_2(n_2) \ldots p_K(n_K) \]
Example (contd.)

- Stationary distribution:
  \[ p(n_1, n_2) = \left(1 - \frac{\lambda_1}{\mu_1}\right) \left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \left(1 - \frac{\lambda_2}{\mu_2}\right) \left(\frac{\lambda_2}{\mu_2}\right)^{n_2} \]

- Detailed Balance Equations:
  \[
  \begin{align*}
  \mu_1 p(n_1 + 1, n_2) &= \lambda_1 p(n_1, n_2) \\
  \mu_2 p(n_1, n_2 + 1) &= \lambda_2 p(n_1, n_2)
  \end{align*}
  \]

- Verify that the Markov chain is reversible – Kolmogorov criterion
Truncation of a Reversible Markov Chain

- **Theorem 9:** \( \{X(t)\} \) reversible Markov process with state space \( S \), and stationary distribution \( \{p_j; j \in S\} \). Truncated to a set \( E \subset S \), such that the resulting chain \( \{Y(t)\} \) is *irreducible*. Then, \( \{Y(t)\} \) is reversible and has stationary distribution:

\[
\tilde{p}_j = \frac{p_j}{\sum_{k \in E} p_k}, \quad j \in E
\]

- **Remark:** This is the conditional probability that, in steady-state, the original process is at state \( j \), given that it is somewhere in \( E \)

- **Proof:** Verify that:

\[
\tilde{p}_j q_{ji} = \tilde{p}_i q_{ij} \Leftrightarrow \sum_{k \in E} \frac{p_j}{p_k} q_{ji} = \sum_{k \in E} \frac{p_i}{p_k} q_{ij} \Leftrightarrow \sum_{k \in E} p_{ji} = \sum_{k \in E} p_{ij}, \quad i, j \in S; i \neq j
\]

\[
\sum_{j \in E} \tilde{p}_j = \sum_{j \in E} \sum_{k \in E} \frac{p_j}{p_k} = 1
\]
Example: Two Queues with Joint Buffer

- The two independent M/M/1 queues of the previous example share a common buffer of size $B$ – arrival that finds $B$ customers waiting is blocked.
- State space restricted to:
  \[ E = \{(n_1, n_2) : (n_1 - 1)^+ + (n_2 - 1)^+ \leq B\} \]
- Distribution of truncated chain:
  \[ p(n_1, n_2) = p(0, 0) \cdot \rho_1^{n_1} \rho_2^{n_2}, \ (n_1, n_2) \in E \]
- Normalizing:
  \[ p(0, 0) = \left[ \sum_{(n_1, n_2) \in E} \rho_1^{n_1} \rho_2^{n_2} \right]^{-1} \]
- Theorem specifies joint distribution up to the normalization constant.
- Calculation of normalization constant is often tedious.

![State Diagram for $B = 2$]
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Birth-death process

- \( \{X(t)\} \) birth-death process with stationary distribution \( \{p_j\} \)
- Arrival epochs: points of increase for \( \{X(t)\} \)
  - Departure epoch: points of decrease for \( \{X(t)\} \)
- \( \{X(t)\} \) completely determines the corresponding arrival and departure processes
Forward & reversed chains of M/M/* queues

- Poisson arrival process: $\lambda_j = \lambda$, for all $j$
  - Birth-death process called a $(\lambda, \mu_j)$-process
  - Examples: M/M/1, M/M/c, M/M/$\infty$ queues
- Poisson arrivals $\rightarrow$ LAA: for any time $t$, future arrivals are independent of $\{X(s): s \leq t\}$

- $(\lambda, \mu_j)$-process at steady state is reversible: forward and reversed chains are stochastically identical
- $\Rightarrow$ Arrival processes of the forward and reversed chains are stochastically identical
  - $\Rightarrow$ Arrival process of the reversed chain is Poisson with rate $\lambda$
  - $+ \text{ “the arrival epochs of the reversed chain are the departure epochs of the forward chain” } \Rightarrow \textbf{Departure process of the forward chain is Poisson with rate } \lambda$
Forward & reversed chains of M/M/*/ queues (contd.)

- Reversed chain: arrivals after time $t$ are independent of the chain history up to time $t$ (LAA)
- $\Rightarrow$ Forward chain: departures prior to time $t$ and future of the chain $\{X(s): s \geq t\}$ are independent
Burke’s Theorem

**Theorem 10**: Consider an M/M/1, M/M/c, or M/M/∞ system with arrival rate $\lambda$. *Suppose that the system starts at steady-state*. Then:

1. The departure process is Poisson with rate $\lambda$
2. At each time $t$, the number of customers in the system is independent of the departure times prior to $t$

**Fundamental result for study of networks of M/M/* queues**, where output process from one queue is the input process of another
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Single-Server Queues in Tandem

Customers arrive at queue 1 according to Poisson process with rate $\lambda$.

Service times exponential with mean $1/\mu_i$. Assume service times of a customer in the two queues are independent.

Assume $\rho_i = \lambda/\mu_i < 1$

What is the joint stationary distribution of $N_1$ and $N_2$ – number of customers in each queue?

Result: in steady state the queues are independent and

$$p(n_1, n_2) = (1 - \rho_1) \rho_1^{n_1} \cdot (1 - \rho_2) \rho_2^{n_2} = p_1(n_1) \cdot p_2(n_2)$$
Single-Server Queues in Tandem

- Q1 is a M/M/1 queue. At steady state its departure process is Poisson with rate $\lambda$. Thus Q2 is also M/M/1.

- Marginal stationary distributions:
  \[ p_1(n_1) = (1 - \rho_1) \rho_1^{n_1}, \quad n_1 = 0,1,... \quad p_2(n_2) = (1 - \rho_2) \rho_2^{n_2}, \quad n_2 = 0,1,... \]

- To complete the proof: establish independence at steady state

- Q1 at steady state: at time $t$, $N_1(t)$ is independent of departures prior to $t$, which are arrivals at Q2 up to $t$. Thus $N_1(t)$ and $N_2(t)$ independent:
  \[ P\{N_1(t) = n_1, N_2(t) = n_2\} = P\{N_1(t) = n_1\}P\{N_2(t) = n_2\} = p_1(n_1) \cdot P\{N_2(t) = n_2\} \]

- Letting $t \rightarrow \infty$, the joint stationary distribution
  \[ p(n_1, n_2) = p_1(n_1) \cdot p_2(n_2) = (1 - \rho_1) \rho_1^{n_1} \cdot (1 - \rho_2) \rho_2^{n_2} \]
Queues in Tandem

- **Theorem 11**: Network consisting of \( K \) single-server queues in tandem. Service times at queue \( i \) exponential with rate \( \mu_i \), independent of service times at any queue \( j \neq i \). Arrivals at the first queue are Poisson with rate \( \lambda \). The stationary distribution of the network is:
  \[
p(n_1, \ldots, n_K) = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^{n_i}, \quad n_i = 0, 1, \ldots; \quad i = 1, \ldots, K
  \]

- At *steady state* the queues are independent; the distribution of queue \( i \) is that of an isolated M/M/1 queue with arrival and service rates \( \lambda \) and \( \mu_i \):
  \[
p_i(n_i) = (1 - \rho_i) \rho_i^{n_i}, \quad n_i = 0, 1, \ldots
  \]

- Are the queues independent if not in steady state? Are stochastic processes \( \{N_1(t)\} \) and \( \{N_2(t)\} \) independent?
Queues in Tandem: State-Dependent Service Rates

- **Theorem 12:** Network consisting of $K$ queues in tandem. Service times at queue $i$ exponential with rate $\mu_i(n_i)$ when there are $n_i$ customers in the queue – independent of service times at any queue $j \neq i$. Arrivals at the first queue are Poisson with rate $\lambda$. The stationary distribution of the network is:

$$p(n_1, \ldots, n_K) = \prod_{i=1}^{K} p_i(n_i), \quad n_i = 0, 1, \ldots; \quad i = 1, \ldots, K$$

where $\{p_i(n_i)\}$ is the stationary distribution of queue $i$ in isolation with Poisson arrivals with rate $\lambda$.

- **Examples:** $/M/c$ and $/M/\infty$ queues
  - If queue $i$ is $/M/\infty$, then:

$$p_i(n_i) = \frac{(\lambda/\mu_i)^{n_i}}{n_i!} e^{-\lambda/\mu_i}, \quad n_i = 0, 1, \ldots$$
Summary

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
  - Multi-dimensional Markov chains
- Burke’s Theorem
- Queues in Tandem