Queuing Analysis:

Review of Markov Chain Theory

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Outline

- Markov Chain
- Discrete-Time Markov Chains
- Calculating Stationary Distribution
- Global Balance Equations
- Birth-Death Process
  - Detailed Balance Equations
- Generalized Markov Chains
- Continuous-Time Markov Chains
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Markov Chain?

- Stochastic process that takes values in a *countable* set
  - Example: \(\{0,1,2,...,m\}\), or \(\{0,1,2,...\}\)
  - Elements represent possible “states”
  - Chain transits from state to state

- *Memoryless (Markov) Property*: Given the present state, future transitions of the chain are independent of past history

- Markov Chains: discrete- or continuous- time
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Discrete-Time Markov Chain

- Discrete-time stochastic process \( \{X_n: n = 0, 1, 2, \ldots \} \)
- Takes values in \( \{0, 1, 2, \ldots \} \)
- Memoryless property:
  \[
P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0 \} = P\{X_{n+1} = j \mid X_n = i \}
  \]
  \[
P_{ij} = P\{X_{n+1} = j \mid X_n = i \}
  \]
- Transition probabilities \( P_{ij} \)
  \[
P_{ij} \geq 0, \quad \sum_{j=0}^{\infty} P_{ij} = 1
  \]
- Transition probability matrix \( P = [P_{ij}] \)
Chapman-Kolmogorov Equations

- $n$ step transition probabilities

\[ P_{ij}^n = P\{X_{n+m} = j \mid X_m = i\}, \quad n, m \geq 0, \ i, j \geq 0 \]

- How to calculate?

  - Chapman-Kolmogorov equations

\[ P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m, \quad n, m \geq 0, \ i, j \geq 0 \]

  - $P_{ij}^n$ is element $(i, j)$ in matrix $P^n$

  - Recursive computation of state probabilities
State Probabilities – Stationary Distribution

- State probabilities (time-dependent)
  \[ \pi^n_j = P\{X_n = j\}, \quad \pi^n = (\pi^n_0, \pi^n_1, \ldots) \]
  \[ P\{X_n = j\} = \sum_{i=0}^{\infty} P\{X_{n-1} = i\}P\{X_n = j \mid X_{n-1} = i\} \implies \pi^n_j = \sum_{i=0}^{\infty} \pi^{n-1}_i P_{ij} \]

  In matrix form:
  \[ \pi^n = \pi^{n-1}P = \pi^{n-2}P^2 = \ldots = \pi^0 P^n \]

- If time-dependent distribution converges to a limit
  \[ \pi = \lim_{n \to \infty} \pi^n \quad \pi = \pi P \]
  \[ \pi \] is called the *stationary distribution* (or *steady state distribution*)
  - existence depends on the structure of Markov chain
Classification of Markov Chains

Irreducible:
- States i and j communicate:
  \[ \exists n, m: P_{ij}^n > 0, P_{ji}^m > 0 \]
- Irreducible Markov chain: all states communicate

Aperiodic:
- State i is periodic:
  \[ \exists d > 1: P_{ii}^n > 0 \Rightarrow n = \alpha d \]
- Aperiodic Markov chain: none of the states is periodic
Limit Theorems

Theorem 1: Irreducible aperiodic Markov chain

- For every state \( j \), the following limit

\[
\pi_j = \lim_{n \to \infty} P \{ X_n = j \mid X_0 = i \}, \quad i = 0, 1, 2, \ldots
\]

exists and is independent of initial state \( i \)

- \( N_j(k) \): number of visits to state \( j \) up to time \( k \)

\[
P \left\{ \pi_j = \lim_{k \to \infty} \frac{N_j(k)}{k} \mid X_0 = i \right\} = 1
\]

\[\Rightarrow \pi_j: \text{frequency the process visits state } j\]
Existence of Stationary Distribution

**Theorem 2:** Irreducible aperiodic Markov chain. There are two possibilities for scalars:

\[
\pi_j = \lim_{n \to \infty} P\{X_n = j \mid X_0 = i\} = \lim_{n \to \infty} P_{ij}^n
\]

1. \( \pi_j = 0 \), for all states \( j \) ➡ No stationary distribution
2. \( \pi_j > 0 \), for all states \( j \) ➡ \( \pi \) is the *unique* stationary distribution

**Remark:** If the number of states is finite, case 2 is the only possibility.
Ergodic Markov Chains

- A state \( j \) is \textit{positive recurrent} if the process returns to state \( j \) “infinitely often”

- Formal definition:
  - \( F_{ij}(n) \) \((n \geq 1)\): the probability, given \( X_0 = i \), that state \( j \) occurs at some time between 1 and \( n \) inclusive
  - \( T_{ij} \): the first passage time from \( i \) to \( j \)
  - A state \( j \) is \textit{recurrent} (or \textit{persistent}) if \( F_{jj}(\infty) = 1 \), and \textit{transient} otherwise
  - A state \( j \) is \textit{positive recurrent} (or \textit{non-null persistent}) if \( F_{jj}(\infty) = 1 \) and \( E(T_{jj}) < \infty \)
  - A state \( j \) is \textit{null recurrent} (or \textit{null persistent}) if \( F_{jj}(\infty) = 1 \) but \( E(T_{jj}) = \infty \)

- Note: “positive recurrent \( \Rightarrow \) irreducible” always hold, but “irreducible \( \Rightarrow \) positive recurrent” is guaranteed to hold only for finite MC
Ergodic MC (contd.)

- Example: a MC with countably infinite state space

All states are positive recurrent if $p < \frac{1}{2}$, null recurrent if $p = \frac{1}{2}$, and transient if $p > \frac{1}{2}$

- A state is ergodic if it is aperiodic and positive recurrent

- A MC is ergodic if every state is ergodic

- Ergodic chains have a unique stationary distribution

$$\pi_j = 1/E(T_{jj}), \ j = 0, 1, 2, ...$$

- Note: Ergodicity $\Rightarrow$ Time Averages $=$ Stochastic Averages
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- **Calculating Stationary Distribution**
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Calculation of Stationary Distribution

A. Finite number of states

- Solve explicitly the system of equations

\[ \pi_j = \sum_{i=0}^{m} \pi_i P_{ij}, \quad j = 0,1,\ldots,m \]

\[ \sum_{i=0}^{m} \pi_i = 1 \]

- Or, numerically from \( P^n \) which converges to a matrix with rows equal to \( \pi \)
  - Suitable for a small number of states

B. Infinite number of states

- Cannot apply previous methods to problem of infinite dimension

- Guess a solution to recurrence:

\[ \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0,1,\ldots, \]

\[ \sum_{i=0}^{\infty} \pi_i = 1 \]
Example: Finite Markov Chain

- Absent-minded professor uses two umbrellas when commuting between home and office.
- If it rains and an umbrella is available at her location, she takes it. If it does not rain, she always forgets to take an umbrella.
- Let $p$ be the probability of rain each time she commutes.

**Q:** What is the probability that she gets wet on any given day?

- Markov chain formulation
- $i$ is the number of umbrellas available at her current location

![Transition matrix diagram]

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$
Example: Finite Markov Chain

\[ P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} \]

\[ \begin{aligned} \pi_0 &= (1-p)\pi_2 \\ \pi_1 &= (1-p)\pi_1 + p\pi_2 \\ \pi_2 &= \pi_0 + p\pi_1 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned} \]

\[ P\{\text{gets wet}\} = \pi_0 p = p \frac{1-p}{3-p} \]
Example: Finite Markov Chain

Taking $p = 0.1$:

$$
\pi = \left( \frac{1-p}{3-p}, \frac{1}{3-p}, \frac{1}{3-p} \right) = (0.310, 0.345, 0.345)
$$

$$
P = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0.9 & 0.1 \\
0.9 & 0.1 & 0
\end{bmatrix}
$$

Numerically determine limit of $P^n$

$$
\lim_{n \to \infty} P^n = \begin{bmatrix}
0.310 & 0.345 & 0.345 \\
0.310 & 0.345 & 0.345 \\
0.310 & 0.345 & 0.345
\end{bmatrix} \\
\text{ (n \approx 150)}
$$

Effectiveness depends on structure of $P$
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Global Balance Equations

- **Global Balance Equations (GBE)**

\[
\pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \iff \pi_j \sum_{i \neq j} P_{ji} = \sum_{i \neq j} \pi_i P_{ij}, \quad j \geq 0
\]

- \( \pi_j P_{ji} \) is the frequency of transitions from \( j \) to \( i \)

\[
\left( \begin{array}{c}
\text{Frequency of} \\
\text{transitions out of} \ j
\end{array} \right) = \left( \begin{array}{c}
\text{Frequency of} \\
\text{transitions into} \ j
\end{array} \right)
\]

- **Intuition:** 1) \( j \) visited infinitely often; 2) for each transition out of \( j \) there must be a subsequent transition into \( j \) with probability 1
Global Balance Equations (contd.)

- **Alternative Form of GBE**
  \[
  \sum_{j \in S} \pi_j \sum_{i \in S} P_{ji} = \sum_{i \in S} \pi_i \sum_{j \in S} P_{ij}, \quad S \subseteq \{0,1,2,\ldots\}
  \]

- **If a probability distribution satisfies the GBE, then it is the unique stationary distribution of the Markov chain**

- **Finding the stationary distribution:**
  - Guess distribution from properties of the system
  - Verify that it satisfies the GBE
  - Special structure of the Markov chain simplifies task
Global Balance Equations – Proof

First form:
\[ \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad \text{and} \quad \sum_{i=0}^{\infty} P_{ji} = 1 \implies \]
\[ \pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \iff \pi_j \sum_{i \neq j} P_{ji} = \sum_{i \neq j} \pi_i P_{ij} \]

Second form:
\[ \pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \implies \sum_{j \in S} \pi_j \left( \sum_{i \in S} P_{ji} + \sum_{i \notin S} P_{ji} \right) = \sum_{j \in S} \left( \sum_{i \in S} \pi_i P_{ij} + \sum_{i \notin S} \pi_i P_{ij} \right) \implies \]
\[ \sum_{j \in S} \pi_j \sum_{i \notin S} P_{ji} = \sum_{i \in S} \sum_{j \in S} P_{ij} \]
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Birth-Death Process

- One-dimensional Markov chain with transitions only between neighboring states: $P_{ij}=0$, if $|i-j|>1$

- Detailed Balance Equations (DBE)

$$\pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n} \quad n = 0,1,...$$

- Proof: GBE with $S=\{0,1,...,n\}$ give:

$$\sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_j P_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_i P_{ij} \Rightarrow \pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n}$$
Example: Discrete-Time Queue

- In a time-slot, one packet arrival with probability $p$ or zero arrivals with probability $1-p$
- In a time-slot, the packet in service departs with probability $q$ or stays with probability $1-q$
- Independent arrivals and service times
- State: number of packets in system

\[
\begin{align*}
0 & \quad p & 1 & \quad p(1-q) & 2 & \ldots & n & \quad p & \quad p(1-q) & n+1 & \ldots \\
(1-p) & q(1-p) & (1-p)(1-q) + pq & q(1-p) & & & q(1-p) & (1-p)(1-q) + pq & \\
\end{align*}
\]
Example: Discrete-Time Queue (contd.)

\[ \pi_0 p = \pi_1 q(1-p) \Rightarrow \pi_1 = \frac{p/q}{1-p} \pi_0 \]

\[ \pi_n p(1-q) = \pi_{n+1} q(1-p) \Rightarrow \pi_{n+1} = \frac{p(1-q)}{q(1-p)} \pi_n, \quad n \geq 1 \]

Define: \( \rho \equiv \frac{p}{q}, \quad \alpha \equiv \frac{p(1-q)}{q(1-p)} \)

\[
\begin{cases}
\pi_1 = \frac{\rho}{1-p} \pi_0 \\
\pi_n = \alpha^{n-1} \frac{\rho}{1-p} \pi_0, \quad n \geq 1 \\
\pi_{n+1} = \alpha \pi_n, \quad n \geq 1
\end{cases}
\]
Example: Discrete-Time Queue (contd.)

- Having determined the distribution as a function of $\pi_0$
  $$\pi_n = \alpha^{n-1} \frac{\rho}{1-p} \pi_0, \ n \geq 1$$
  How to calculate the normalization constant $\pi_0$?

- Probability conservation law:
  $$\sum_{n=0}^{\infty} \pi_n = 1 \Rightarrow \pi_0 = \left[ 1 + \frac{\rho}{1-p} \sum_{n=1}^{\infty} \alpha^{n-1} \right]^{-1} = \left[ 1 + \frac{\rho}{(1-p)(1-\alpha)} \right]^{-1}$$

- Noting that
  $$(1-p)(1-\alpha) = (1-p) \frac{q(1-p) - p(1-q)}{q(1-p)} = \frac{q-p}{q} = 1-\rho$$

  $$\begin{cases} 
  \pi_0 = 1-\rho \\
  \pi_n = \rho(1-\alpha)\alpha^{n-1}, \ n \geq 1 
  \end{cases}$$
Detailed Balance Equations

- General case:
  \[ \pi_j P_{ji} = \pi_i P_{ij} \quad i, j = 0,1,... \]

- Need NOT hold for every Markov chain

- If hold, it implies the GBE; greatly simplify the calculation of stationary distribution

Methodology:

- Assume DBE hold – have to guess their form
- Solve the system defined by DBE and \( \Sigma_i \pi_i = 1 \)
  - If system is inconsistent, then DBE does not hold
  - If system has a solution \( \{\pi_i: i=0,1,...\} \), then it is the unique stationary distribution
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Generalized Markov Chains

- Markov chain on a set of states \{0, 1, \ldots\}, that whenever enters state \(i\)
  - The next state that will be entered is \(j\) with probability \(P_{ij}\)
  - Given that the next state entered will be \(j\), the time it spends at state \(i\) until the transition occurs is a RV with distribution \(F_{ij}\)

- \(\{Z(t): t \geq 0\}\) describing the state of the chain at time \(t\): *Generalized Markov chain*, or *Semi-Markov process*

  - Does GMC have the Markov property?
    - Future depends on 1) the present state, and 2) the length of time the process has spent in this state
Generalized Markov Chains (contd.)

- $T_i$: time process spends at state $i$, before making a transition – *holding time*

- Probability distribution function of $T_i$

  \[
  H_i(t) = P\{T_i \leq t\} = \sum_{j=0}^{\infty} P\{T_i \leq t \mid \text{next state } j\} P_{ij} = \sum_{j=0}^{\infty} F_{ij}(t) P_{ij}
  \]

  \[
  E[T_i] = \int_0^\infty t \, dH_i(t)
  \]

- $T_{ii}$: time between successive transitions to $i$

- $X_n$ is the $n^{th}$ state visited. \{\(X_n: n=0,1,\ldots\}\)
  - Is a Markov chain: *embedded* Markov chain
  - Has transition probabilities $P_{ij}$

- Semi-Markov process *irreducible*: if its embedded Markov chain is irreducible
Limit Theorems

Theorem 3: given an irreducible semi-Markov process w/ $E[T_{ii}] < \infty$

- For any state $j$, the following limit
  \[
  p_j = \lim_{t \to \infty} P\{Z(t) = j \mid Z(0) = i\}, \quad i = 0,1,2,\ldots
  \]
  exists and is independent of the initial state.
  \[
  p_j = \frac{E[T_{ij}]}{E[T_{ii}]}
  \]

- $T_j(t)$: time spent at state $j$ up to time $t$
  \[
  P\left\{ p_j = \lim_{t \to \infty} \frac{T_j(t)}{t} \mid Z(0) = i \right\} = 1
  \]
  - $p_j$ is equal to the proportion of time spent at state $j$
Theorem 4: given an irreducible semi-Markov process where $E[T_{ii}] < \infty$, and the embedded Markov chain is ergodic w/ stationary distribution $\pi$

$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \ j \geq 0; \ \sum_{i=0}^{\infty} \pi_i = 1$

then, with probability 1, the occupancy distribution of the semi-Markov process

$p_j = \frac{\pi_j E[T_j]}{\sum_i \pi_i E[T_i]}, \ j = 0, 1, ...$

- $\pi_j$: proportion of transitions into state $j$
- $E[T_j]$: mean time spent at $j$

► Probability of being at $j$ is proportional to $\pi_j E[T_j]$
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Continuous-Time Markov Chains (def.?)

Continuous-time process \( \{X(t): t \geq 0\} \) taking values in \( \{0,1,2,...\} \).

Whenever it enters state \( i \)

- Time it spends at state \( i \) is *exponentially distributed* with parameter \( \nu_i \)

- When it leaves state \( i \), it enters state \( j \) with probability \( P_{ij} \) where \( \sum_{j \neq i} P_{ij} = 1 \)

- Continuous-time Markov chain is a semi-Markov process with

\[
F_{ij}(t) = 1 - e^{-\nu_i t}, \quad i, j = 0,1,...
\]

- Exponential holding time \( \Rightarrow \) a continuous-time Markov chain has the Markov property
Continuous-Time Markov Chains

- When at state $i$, the process makes transitions to state $j \neq i$ with rate:
  \[ q_{ij} \equiv \nu_i P_{ij} \]

- Total rate of transitions out of state $i$
  \[ \sum_{j \neq i} q_{ij} = \nu_i \sum_{j \neq i} P_{ij} = \nu_i \]

- Average time spent at state $i$ before making a transition:
  \[ E[T_i] = 1/\nu_i \]
Occupancy Probability

- A continuous-time Markov chain is irreducible and regular, if
  - Embedded Markov chain is irreducible
  - Number of transitions in a finite time interval is finite with probability 1
- From Theorem 3: for any state $j$, the limit
  
  \[ p_j = \lim_{t \to \infty} P\{X(t) = j \mid X(0) = i\}, \quad i = 0, 1, 2, \ldots \]

  exists and is independent of the initial state
  - $p_j$ is the steady-state occupancy probability of state $j$
  - $p_j$ is equal to the proportion of time spent at state $j$
Global Balance Equations

- Two possibilities for the occupancy probabilities:
  - \( p_j = 0 \), for all \( j \)
  - \( p_j > 0 \), for all \( j \), and \( \sum_j p_j = 1 \)

- Global Balance Equations
  \[
  p_j \sum_{i \neq j} q_{ji} = \sum_{i \neq j} p_i q_{ij}, \quad j = 0, 1, \ldots
  \]
  - Rate of transitions out of \( j \) = rate of transitions into \( j \)
  - If a distribution \( \{p_j: j = 0, 1, \ldots\} \) satisfies GBE, then it is the unique occupancy distribution of the Markov chain

- Alternative form of GBE:
  \[
  \sum_{j \in S} p_j \sum_{i \in S} q_{ji} = \sum_{i \in S} p_i \sum_{j \in S} q_{ij}, \quad S \subseteq \{0, 1, \ldots\}
  \]
Detailed Balance Equations

- Detailed Balance Equations
  \[ p_j q_{ji} = p_i q_{ij}, \quad i, j = 0, 1, ... \]

😊 Simplify the calculation of the stationary distribution

😢 Need not hold for every Markov chain

- Examples: birth-death processes, and reversible Markov chains
Birth-Death Process

- Transitions only between neighboring states
  \[ q_{i,i+1} = \lambda_i, \quad q_{i,i-1} = \mu_i, \quad q_{ij} = 0, \quad |i-j| > 1 \]

- Detailed Balance Equations
  \[ \lambda_n p_n = \mu_{n+1} p_{n+1}, \quad n = 0, 1, \ldots \]

- Proof: GBE with \( S =\{0,1,\ldots,n\} \) give:
  \[ \sum_{j=0}^{n} \sum_{i=0}^{\infty} p_j q_{ji} = \sum_{j=0}^{n} \sum_{i=0}^{\infty} p_i q_{ij} \Rightarrow \lambda_n p_n = \mu_{n+1} p_{n+1} \]
Birth-Death Process

\[ \mu_n p_n = \lambda_{n-1} p_{n-1} \Rightarrow \]
\[ p_n = \frac{\lambda_{n-1}}{\mu_n} p_{n-1} = \frac{\lambda_{n-1}}{\mu_n} \frac{\lambda_{n-2}}{\mu_{n-1}} p_{n-2} = \cdots = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} p_0 \]
\[ p_0 = p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \]

\[ \sum_{n=0}^{\infty} p_n = 1 \iff p_0 \left[ 1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \right] = 1 \iff p_0 = \left[ 1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \right]^{-1}, \text{ if } \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} < \infty \]

- Use DBE to determine state probabilities as a function of \( p_0 \)
- Use the probability conservation law to find \( p_0 \)
- Using DBE in solving problems:
  - Prove that DBE hold, or
  - Justify validity (e.g. reversible process), or
  - Assume they hold – have to guess their form – and solve system
M/M/1 Queue

- Arrival process: Poisson with rate $\lambda$
- Service times: iid, exponential with parameter $\mu$
- Service times and interarrival times: independent
- Single server
- Infinite waiting room
- $\mathbb{N}(t)$: Number of customers in system at time $t$ (state)
M/M/1 Queue

- Birth-death process → DBE
  \[ \mu p_n = \lambda p_{n-1} \Rightarrow \]
  \[ p_n = \frac{\lambda}{\mu} p_{n-1} = \rho p_{n-1} = \ldots = \rho^n p_0 \]

- Normalization constant
  \[ \sum_{n=0}^{\infty} p_n = 1 \Leftrightarrow p_0 \left[ 1 + \sum_{n=1}^{\infty} \rho^n \right] = 1 \Leftrightarrow p_0 = 1 - \rho \quad \text{if } \rho < 1 \]

- Stationary distribution
  \[ p_n = \rho^n (1 - \rho), \quad n = 0, 1, \ldots \]
The M/M/1 Queue

- **Average number of customers**
  \[ N = \sum_{n=0}^{\infty} np_n = (1 - \rho) \sum_{n=0}^{\infty} n \rho^n = (1 - \rho) \rho \sum_{n=0}^{\infty} n \rho^{n-1} \]
  \[ \Rightarrow N = \rho(1 - \rho) \frac{1}{(1 - \rho)^2} = \frac{\rho}{1 - \rho} = \frac{\lambda}{\mu - \lambda} \]

- **Applying Little’s Theorem, we have**
  \[ T = \frac{N}{\lambda} = \frac{1}{\frac{\lambda}{\mu - \lambda}} = \frac{1}{\mu - \lambda} \]

- **Similarly, the average waiting time and number of customers in the queue is given by**
  \[ W = T - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda} \text{ and } N_Q = \lambda W = \frac{\rho^2}{1 - \rho} \]
Summary

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Homework #8

- Problem 3.14 of R1

- Hints:
  - For a service system, the expected number of customers is finite if the service rate is greater than the customer arrival rate.
  - To solve the problem, think of how to model the system as a Markov process. You may also find Little's Theorem be of some use in solving the problem.

- Grading:
  - Overall points 100
    - 30 points for 3.14(a)
    - 70 points for 3.14(b)