Chapter 4:

1. Exercise 4.4:

4.4 A process $A$ with envelope $E_1$ is passed through a leaky bucket regulator with envelope $E_2$. Show that the resulting output has envelope $E_1 \ast E_2$.

**Solution:** Since the regulator is a leaky bucket, we have $D = A \ast E_2$. Since $A$ has envelope $E_1$, we further have $A \leq A \ast E_1$. Combining these observations, we have $D \leq (A \ast E_1) \ast E_2$ (in doing which, we have already used the property that $\ast$ distributes over min). Now use the associativity of $\ast$ and the fact that $E_2 = E_2 \ast E_2$ to obtain $D \leq (A \ast E_2) \ast (E_1 \ast E_2) = D \ast (E_1 \ast E_2)$, i.e., $D$ has envelope $E_1 \ast E_2$.

2. Exercise 4.5:

4.5 Consider a source with peak rate $R$ and packet size $L$, i.e., the packets are spaced by no less than $\frac{L}{R}$ seconds (e.g., a voice coder and packetiser emit 200 byte packets (160 bytes of PCM voice plus 40 bytes of RTP/UDP/IP headers) every 20ms, yielding $L = 200$ bytes and $R = 10$KBps). This source is shaped with a LB with parameters $(\sigma, \rho)$, with $\sigma \geq L$, and $\rho \leq R$. The source is served by a latency rate server with lower service curve that has rate $r$ and latency $d$ (or, a tandem of service elements that has this lower service curve). Show that:

a. the source has an envelope $(L + Rt)I_{\{t \geq 0\}}$,

b. the output of the LB has an envelope $\min((L + Rt)I_{\{t \geq 0\}},(\sigma + \rho t)I_{\{t \geq 0\}})$, and
c. for \( r < R \), the delay experienced by the source is bounded by

\[
d + \left( \frac{\sigma - L}{r} \right) \left( \frac{R - r}{R - \rho} \right) + \frac{L}{r}
\]

**Solution:**

a. Consider an interval \( (t, t + \tau) \), with \( \tau \geq 0 \). Suppose a packet arrives at \( t \); no more than \( \frac{\tau}{(L/R)} \) additional packets can arrive over the remaining interval. Hence the amount of data emitted by the source over this interval is bounded by \( L + L \times \frac{\tau}{(L/R)} = L + R\tau \).

b. Let us denote the source envelope by \( A(t) \) and the LB envelope by \( E(t) \). It then follows from Exercise 4.4 that the regulated output, \( D(t) \), has envelope \( (A \ast E)(t) \). Now we have

\[
(A \ast E)(t) = \inf_{\tau \in \mathbb{R}} f_\tau(t)
\]

where, for all \( t \in \mathbb{R} \),

\[
f_\tau(t) = \begin{cases} 
(\sigma + \rho(t - \tau))I_{(t-\tau \geq 0)} & \tau < 0 \\
(L + R\tau) + (\sigma + \rho(t - \tau))I_{(t-\tau \geq 0)} & \tau \geq 0
\end{cases}
\]
It is then seen that $\inf_{\tau < 0} f_\tau(t) = (\sigma + \rho(t))I_{\{t \geq 0\}}$. For $\tau \geq 0$, consider the two cases: (i) $\tau \leq t$, for which we have

$$f_\tau(t) = (L + R\tau) + (\sigma + \rho(t - \tau))$$

$$\geq \sigma + \rho t$$

since $R > \rho$, and (ii) $\tau > t$, for which we have

$$f_\tau(t) = L + R\tau$$

$$\geq L + R t$$

It follows that

$$(A \ast E)(t) \geq \min((L + R t)I_{\{t \geq 0\}}, (\sigma + \rho t)I_{\{t \geq 0\}})$$

We already know that, for causal $A$ and $E$, $(A \ast E)(t) \leq \min\{A(t), E(t)\}$. Hence we conclude that

$$(A \ast E)(t) = \min((L + R t)I_{\{t \geq 0\}}, (\sigma + \rho t)I_{\{t \geq 0\}})$$

Figure 4.1: Answer to the third part of Exercise 4.5

c. The solution can be obtained geometrically and is clear from Figure 4.1